Hyperbolic Components

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With an Appendix by A. Poirier.

Abstract. Consider polynomial maps $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$, or more generally polynomial maps from a finite union of copies of $\mathbb{C}$ to itself. In the space of suitably normalized maps of this type, the hyperbolic maps form an open set called the ***hyperbolic locus***. The various connected components of this hyperbolic locus are called ***hyperbolic components***, and those hyperbolic components with compact closure (or equivalently those contained in the "connectedness locus") are called ***bounded*** hyperbolic components. It is shown that each bounded hyperbolic component is a topological cell containing a unique post-critically finite map called its ***center point***. For each degree d, the bounded hyperbolic components can be separated into finitely many distinct types, each of which is characterized by a suitable ***reduced mapping scheme*** $\overline{S}_f$. Any two components with the same reduced mapping scheme are canonically biholomorphic to each other. There are similar statements for real polynomial maps, for polynomial maps with marked critical points, and for rational maps. Appendix A, by Alfredo Poirier, proves that every reduced mapping scheme can be represented by some classical hyperbolic component, made up of polynomial maps of $\mathbb{C}$. This paper is a revised version of [**M2**], which was circulated but not published in 1992.

1. Introduction.

Definition 1.1. A ***hyperbolic mapping scheme*** S (or briefly a ***scheme***) consists of a finite set $|S|$ of "vertices", together with a map $F = F_S : |S| \to |S|$, and an integer valued ***critical weight function*** $s \mapsto \mathbf{w}(s) \geq 0$, satisfying two conditions:

- Any vertex of weight zero is the iterated forward image of some vertex of positive weight, $\mathbf{w}(s) \geq 1$.
- (Hyperbolicity.) Every periodic orbit under F contains at least one vertex of positive weight.

The number $d(s) = \mathbf{w}(s) + 1 \geq 1$ will be called the ***degree*** of the vertex s. The scheme is called ***reduced*** if $\mathbf{w}(s) \geq 1$ (or $d(s) \geq 2$) for every $s \in |S|$.

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First consider a polynomial map $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ with connected Julia set which is ***hyperbolic*** in the sense that every critical orbit converges to an attracting cycle.

Definition 1.2. The ***full mapping scheme*** S_f of such a map has one vertex s_U corresponding to each component U of the Fatou set which contains a critical or post-critical point. The weight $\mathbf{w}(s_U) \geq 0$ is defined to be the number of critical points in U, counted with multiplicity, and the associated map $F_f : |S_f| \to |S_f|$ carries s_U to $s_{f(U)}$.

However for many purposes a slightly simpler structure is more useful. Every mapping scheme can be simplified to an associated reduced scheme (see Remark 3.1). In particular:

Definition 1.3. The ***reduced mapping scheme*** $\overline{S} = \overline{S}_f$ associated with a hyperbolic polynomial map f can be described as follows:

- There is one vertex $s = s_U \in |\overline{S}|$ for each Fatou component $U \subset K(f)$ which contains at least one critical point.
- The weight $\mathbf{w}(s)$ is again the number of critical points in U, counted with multiplicity.
- The map $F : |\overline{S}| \to |\overline{S}|$ is defined by $F(s_U) = s_{U'}$, where $U' = f^{\circ n}(U)$, $n > 0$, is the first forward image which contains a critical point.

Poirier has shown that every reduced mapping scheme can be obtained in this way, from some hyperbolic map from $\mathbb{C}$ to itself. (See Appendix A.)

Outline of what follows. Section 2 will introduce the space $\mathcal{P}^{S_0}$ of suitably normalized polynomial maps, associated with any mapping scheme S_0, and modify several classical definitions so that they apply in this more general context. Section 3 will provide a graphical description of mapping schemes, and discuss symmetries. Sections 4 and 5 will provide a universal topological model, based on Blaschke products, for hyperbolic components with a specified reduced mapping scheme, showing that each hyperbolic component is a topological cell with a preferred center point. Section 6 will sharpen this result by providing a universal biholomorphic model. Section 7 discusses analogous results for polynomial mappings with real coefficients, and more generally for ***real forms*** of complex polynomial mappings. Section 8 studies polynomial mappings which have been ***critically marked*** by specifying an ordered list of their critical points. It is shown that all of the principal results carry over to the critically marked case. Section 9 proves analogous results for rational maps.

Appendix A, by Alfredo Poirier, shows that every reduced scheme actually occurs as the scheme $\overline{S}_f$ for some critically finite hyperbolic map $f : \mathbb{C} \to \mathbb{C}$. Appendix B studies the number of distinct reduced schemes with given total weight.

The present work is a fairly straightforward extension of ideas originated by Douady, Hubbard, McMullen, Rees and others, and many of the statements were probably known as folk theorems. I am particularly grateful to Branner and Douady for their considerable help with the earlier version, and to Araceli Bonifant, Adam Epstein, Alfredo Poirier, and Scott Sutherland for their help with the present version.

2. The Affine Parameter Spaces $\mathcal{P}^d$ and $\mathcal{P}^{S_0}$.

First consider the classical case of polynomial maps $f : \mathbb{C} \to \mathbb{C}$. (See for example [**D**], [**DH1**], or [**M4**].)

Definition 2.1. A complex polynomial map

$$f(z) = \sum_{j=0}^{d} a_j z^j$$

will be called ***monic*** and ***centered*** if $a_d = 1$ and $a_{d-1} = 0$. (In the degree one case, by definition, only the identity map is monic and centered.) For $d \geq 2$, let $\mathcal{P}^d$ be the complex $(d-1)$-dimensional affine space consisting of all polynomial maps $f : \mathbb{C} \to \mathbb{C}$ which are monic and centered. For each such f, the ***filled Julia set*** $K(f) \subset \mathbb{C}$ is the union of all bounded orbits, and the ***connectedness locus*** $\mathcal{C}^d \subset \mathcal{P}^d$ is the compact set consisting of all polynomials $f \in \mathcal{P}^d$ for which the filled Julia set is connected, or equivalently contains all critical points. A polynomial or rational map is ***hyperbolic*** if the orbit of every critical point converges to an attracting cycle. (Here convergence to the attracting fixed point at infinity is allowed, although we will not be interested in that case.) The open set consisting of all hyperbolic maps will be denoted by $\mathcal{H}^d \subset \mathcal{P}^d$.

A connected component $H \subset \mathcal{H}^d$ has compact closure if and only if it is contained in $\mathcal{C}^d$, or if and only if every critical orbit converges to a finite attracting cycle. Those connected components of $\mathcal{H}^d$ which are contained in $\mathcal{C}^d$ will be called ***bounded hyperbolic components***. It is not hard to see that all of the maps f in such a bounded hyperbolic component H have isomorphic[1] reduced mapping schemes $\overline{S}_f$, so we can use the alternate notation $\overline{S}_H = \overline{S}_f$. Note that the ***total weight***

$$\mathbf{w}(\overline{S}_H) \;=\; \sum_{s \in |\overline{S}_H|} \mathbf{w}(s) \tag{2.1}$$

associated with each $H \subset \mathcal{H}^d$ is equal to the complex dimension $d-1$ of $\mathcal{H}^d$. The number of isomorphism classes of reduced schemes grows rapidly with the total weight $\mathbf{w}(S)$. (See Table 1 for small values of $\mathbf{w}(S)$, and see Figure 2 for the special case $\mathbf{w}(S) = 2$.) For details, see Appendix B.

According to Poirier, every one of these reduced schemes can be realized by a suitable hyperbolic component in $\mathcal{P}^{\mathbf{w}(S)+1}$.

$\mathbf{w}(S)$	1	2	3	4	5	6
number	1	4	12	42	138	494

Table 1. *The numbers of distinct reduced schemes with* $\mathbf{w}(S) \leq 6$.

In order to obtain a canonical model for hyperbolic components with a specified reduced mapping scheme, we need to extend the concept of polynomial map by allowing maps from some disjoint union of finitely many copies of $\mathbb{C}$ to itself. More explicitly, we will consider the following. Let S_0 be an arbitrary mapping scheme.

[1] We will see in §5 that each such H is simply-connected, so that these isomorphisms are uniquely defined.

We will think of the product $|S_0|\times\mathbb{C}$ as a disjoint union of copies of $\mathbb{C}$, indexed by the points $s\in|S_0|$.

Definition 2.2 (The parameter space $\mathcal{P}^{S_0}$). By a ***generalized polynomial map*** based on the scheme S_0 will be meant a map

$$\mathbf{f}:|S_0|\times\mathbb{C}\to|S_0|\times\mathbb{C}$$

which sends each $s\times\mathbb{C}$ onto $F(s)\times\mathbb{C}$ by a polynomial map of degree $d(s)$, where $F=F_{S_0}$. Such a map is **normalized** if each of these polynomial maps $s\times\mathbb{C}\to F(s)\times\mathbb{C}$ is monic and centered. (Compare Remark 3.2.) The complex affine space consisting of all such normalized maps will be denoted by $\mathcal{P}^{S_0}$. Thus $\mathbf{f}\in\mathcal{P}^{S_0}$ if and only if $\mathbf{f}$ has the form

$$\mathbf{f}(s,z) \;=\; \big(F(s),\ f_s(z)\big)\,,$$

where each $f_s:\mathbb{C}\to\mathbb{C}$ is a monic centered polynomial of degree $d(s)=\mathbf{w}(s)+1$. There is a preferred base point $\mathbf{f}_0\in\mathcal{P}^{S_0}$ given by

$$\mathbf{f}_0(s,\,z) \;=\; \big(F(s),\ z^{d(s)}\big)\,. \tag{2.2}$$

In the special case where $|S_0|$ consists of a single point of weight $\mathbf{w}=d-1$, note that $\mathcal{P}^{S_0}$ can be identified with the space $\mathcal{P}^d$ of Definition 2.1. Many of the basic definitions and results in the case of a map $f:\mathbb{C}\to\mathbb{C}$ carry over easily to this more general context. (Proofs will be omitted if they are completely analogous to the proofs in the classical case, as given for example in [**DH1**] or [**M4**].)

Definition 2.3. First consider the "dynamic space" $|S_0|\times\mathbb{C}$. The ***Fatou set*** associated with any map $\mathbf{f}\in\mathcal{P}^{S_0}$ is defined to be the open subset of $|S_0|\times\mathbb{C}$ consisting of all points (s,z) such that the iterates of $\mathbf{f}$, restricted to some neighborhood of (s,z), form a normal family. Each connected component of the Fatou set is called a ***Fatou component***. The map $\mathbf{f}$ is ***hyperbolic*** if every critical orbit converges to a periodic orbit.

There are two reasonable concepts of the "Julia set" in this context. The complement of the Fatou set in $|S_0|\times\mathbb{C}$ will be called the ***fully invariant Julia set*** $J(\mathbf{f})$. Alternatively, following Julia, one could consider the closure of the set of repelling periodic orbits. This forms a compact forward invariant set $J_{\rm rec}(\mathbf{f})$, which can be called the ***recurrent Julia set***. Note that $J_{\rm rec}(\mathbf{f})$ is strictly smaller than $J(\mathbf{f})$ whenever the map $F:|S_0|\to|S_0|$ is not surjective.

The union of all orbits which are bounded (i.e., contained in a compact subset of $|S_0|\times\mathbb{C}$) is a compact set $K(\mathbf{f})\subset|S_0|\times\mathbb{C}$ called the ***filled Julia set***. The boundary $\partial K(\mathbf{f})$ is equal to $J(\mathbf{f})$; and $K(\mathbf{f})$ can be described as the union of $J(\mathbf{f})$ with all bounded Fatou components. Just as in the classical case, every bounded Fatou component is biholomorphic to the open unit disk; and if $\mathbf{f}$ is hyperbolic, then the boundary of each such component is a Jordan curve. (Compare §5.)

Definition 2.4. Now consider the ***parameter space*** $\mathcal{P}^{S_0}$. Evidently $\mathcal{P}^{S_0}$ is a complex affine space with complex dimension equal to the total weight

$$\mathbf{w}(S_0) \;=\; \sum_{s\in|S_0|}\mathbf{w}(s)$$

(or to the total number of critical points, counted with multiplicity). The ***connectedness locus*** is defined to be the compact set $\mathcal{C}^{S_0}\subset\mathcal{P}^{S_0}$ consisting of all maps $\mathbf{f}\in\mathcal{P}^{S_0}$ for which all critical points are contained in $K(\mathbf{f})$. Equivalently, $\mathcal{C}^{S_0}$

can be described as the set of all $\mathbf{f} \in \mathcal{P}^{S_0}$ such that the intersection of $K(\mathbf{f})$ with each $s \times \mathbb{C}$ is connected. The notation $\mathcal{H}^{S_0}$ will be used for the set of all hyperbolic maps in $\mathcal{P}^{S_0}$. Each connected component of $\mathcal{H}^{S_0} \cap \mathcal{C}^{S_0}$ will be called a ***bounded hyperbolic component*** H. Just as in Definition 1.2, we can define the mapping scheme S_H associated with each bounded hyperbolic component H. (In most cases this new scheme S_H will **not** be the same as the ambient scheme S_0, although there is a natural map from S_H onto S_0.)

REMARK 2.5. Both the statement that each hyperbolic component is a topological cell, and the statement that it has a preferred center point, are strongly dependent on the fact that we consider only hyperbolic components within the connectedness locus—the structure of hyperbolic components outside the connectedness locus is very different. For instance, Blanchard, Devaney and Keen [**BDK**] show that the ***shift locus***, the unbounded hyperbolic component consisting of maps for which all critical orbits escape to infinity, has a very complicated fundamental group when $d \geq 3$. (In the somewhat analogous moduli space for quadratic rational maps with marked critical points, there is a similar "shift locus hyperbolic component" which contains a Klein bottle as retract, and hence also has a non-abelian fundamental group. Compare [**M3**, §8.7].)

3. Graphs and Symmetries.

It is often convenient to represent each scheme S by a finite graph $\Gamma(S)$, with the points of $|S|$ as vertices, and with a directed edge leading from each vertex s to $F(s)$. By definition, the **degree** of such an edge is equal to $d(s) = \mathbf{w}(s) + 1$.

In the figures, each vertex of positive critical weight $\mathbf{w} > 0$ is represented by a cluster of $\mathbf{w}$ heavy dots, while vertices of weight zero (if any) are represented by much smaller dots.

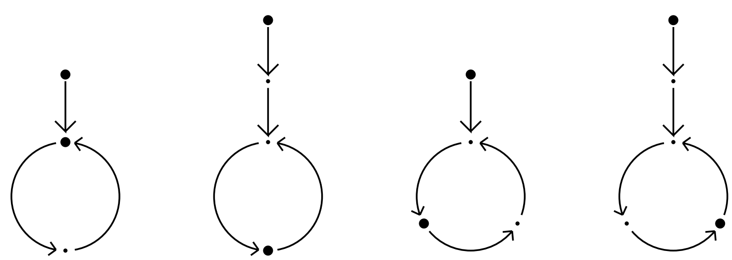


FIGURE 1. *Four different full schemes which give rise to the same reduced scheme.*

REMARK 3.1 (THE ASSOCIATED REDUCED SCHEME). Every mapping scheme S gives rise to an ***associated reduced scheme*** $\overline{S}$, as follows. By definition, $|\overline{S}|$ is the subset of $|S|$ consisting of points of positive weight; and the critical weight of each vertex of $\overline{S}$ is the same as its critical weight in S. The associated function $F_{\overline{S}}$ from $|\overline{S}|$ to itself is obtained by iterating $F_S : |S| \to |S|$ until we reach a vertex s' of positive weight. If we start with the graph $\Gamma(S)$ of an arbitrary mapping scheme, then the graph $\Gamma(\overline{S})$ of the associated reduced scheme can be obtained from $\Gamma(S)$

simply by shrinking each edge of degree one joining s to $F(s)$ to its endpoint $F(s)$. If we start with the full mapping scheme S_f of a hyperbolic map, then clearly the associated reduced scheme $\overline{S}_f$ constructed in this way is identical to the object described in Definition 1.3 above. Note that for any S_0 the affine space $\mathcal{P}^{S_0}$ can be identified with $\mathcal{P}^{\overline{S}_0}$.

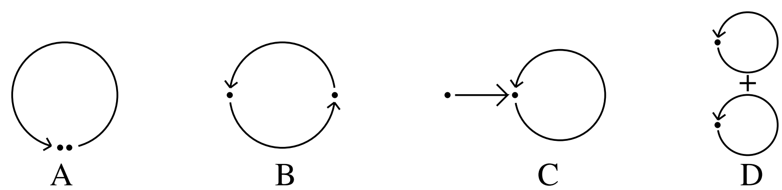


FIGURE 2. *Graphs for the four reduced schemes of weight* $\mathbf{w}(S) = 2$. *(Compare the discussion of cubic polynomials of Example 7.11.)*

The full scheme provides quite a bit of information about any given hyperbolic map which is lost in this reduced scheme. In fact there are infinitely many possible full schemes for each reduced scheme. Figure 1 gives four different examples of schemes of total weight two which are associated with cubic maps of $\mathbb{C}$. All of these correspond to the same reduced scheme, which is shown in Figure 2C.

Figure 2 shows the graphs of all distinct reduced schemes with total weight $\mathbf{w}(S) = 2$. (Compare [**M1**].) The numbers of distinct reduced schemes of given total weight are shown in Table 1 of §2 for $\mathbf{w}(S) \le 6$. (Compare Appendix B.) However, I don't know any formula for the number of such schemes in general.

Symmetries. Let $\mathcal{G} = \mathcal{G}(S)$ be the finite abelian group consisting of all maps $\mathbf{g} : |S| \times \mathbb{C} \to |S| \times \mathbb{C}$ which send each $s \times \mathbb{C}$ linearly onto itself, and which commute with the base map

$$\mathbf{f}_0(s, z) = \big(F(s),\, z^{d(s)}\big)\,.$$

More explicitly, each $\mathbf{g} \in \mathcal{G}$ must have the form

$$\mathbf{g}(s, z) \;=\; (s,\, \rho_s z)\,, \tag{3.1}$$

where each vertex s is assigned a root of unity ρ_s satisfying the condition

$$\rho_s^{d(s)} \;=\; \rho_{F(s)}\,. \tag{3.2}$$

REMARK 3.2. The restriction to monic centered maps in Definition 2.2 can be justified as follows. If we start with an arbitrary generalized polynomial map with scheme S_0 which is not required to be monic or centered, then it is not difficult to find an automorphism $\mathbf{h}$ of $|S_0| \times \mathbb{C}$ which carries each component holomorphically onto itself, so that $\mathbf{h}^{-1} \circ \mathbf{f} \circ \mathbf{h} \in \mathcal{P}^{S_0}$. This $\mathbf{h}$ is uniquely determined up to composition with some $\mathbf{g} \in \mathcal{G}(S_0)$.

The order of this abelian group $\mathcal{G}(S)$ can be computed as follows. It suffices to consider the connected case, since the automorphism group $\mathcal{G}(S + S')$ of a disjoint union is clearly isomorphic to $\mathcal{G}(S) \times \mathcal{G}(S')$.

LEMMA 3.3. *The order of the symmetry group $\mathcal{G}(S)$ of a connected mapping scheme is equal to the product*

$$\big(d_1 \cdots d_k - 1\big)\, d_{k+1} \cdots d_\ell\,,$$

where $d_1, \ldots, d_k$ *are the degrees of the vertices which belong to its unique cycle, and* $d_{k+1}, \ldots, d_\ell$ *are the degrees of the remaining aperiodic vertices.*

PROOF. First consider the case where all vertices are periodic, so that S consists only of the cycle. Then we must have $\rho_j^{d_j} = \rho_{j+1}$, where j is understood to be an integer modulo k. It follows that

$$\rho_j^{d_1 d_2 \cdots d_k} \;=\; \rho_j\,.$$

Thus ρ_1 can be an arbitrary $(d_1 d_2 \cdots d_k - 1)$-th root of unity, and the remaining ρ_j are then uniquely determined. Thus, in this case, $\mathcal{G}$ is cyclic of order $d_1 \cdots d_k - 1$.

The proof now continues inductively, building up the scheme S by adding one new vertex at a time outside of the cycle. Evidently there are exactly d_j possible choices for each new ρ_j, and the conclusion follows. □

In particular, vertices of degree one make no contribution to the order of $\mathcal{G}(S)$; and in fact it is easy to check that the symmetry group $\mathcal{G}(\overline{S})$ for the associated reduced scheme is isomorphic to $\mathcal{G}(S)$.

REMARK 3.4. Each $\mathbf{g} \in \mathcal{G}(S)$ acts linearly on the affine space $\mathcal{P}^S$, sending each map $\mathbf{f} : |S| \times \mathbb{C} \to |S| \times \mathbb{C}$ to the map

$$\mathbf{f}^{\mathbf{g}} \;=\; \mathbf{g}^{-1} \circ \mathbf{f} \circ \mathbf{g}\,.$$

In fact setting $\mathbf{f}(s, z) = \big(F(s),\, z^d + a_{d-2} z^{d-2} + \cdots + a_0\big)$ with $d = d(s)$, it follows easily that

$$\mathbf{f}^{\mathbf{g}}(s,\, z) \;=\; \left(F(s),\, \frac{(\rho_s z)^d \,+\, a_{d-2}(\rho_s z)^{d-2} \,+\, \cdots \,+\, a_1 \rho_s z \,+\, a_0}{\rho_{F(s)}}\right) \tag{3.3}$$

is again an element of $\mathcal{P}^S$. However, this action of $\mathcal{G}(S)$ on $\mathcal{P}^S$ is not always faithful.

DEFINITION 3.5. Let $\mathcal{G}_0(S)$ be the subgroup of $\mathcal{G}(S)$ consisting of all $\mathbf{g} \in \mathcal{G}(S)$ which commute with every $\mathbf{f} \in \mathcal{P}^S$, so that the action of $\mathbf{g}$ on $\mathcal{P}^S$ is trivial. Thus the quotient group $\mathcal{G}(S)/\mathcal{G}_0(S)$ acts faithfully (i.e., effectively) on $\mathcal{P}^S$.

It will be convenient to define the "free" vertices of S to be those which do not belong to the image $F(|S|) \subset |S|$. It follows from Definition 1.1 that every free vertex has degree $d(s) \geq 2$.

LEMMA 3.6. *An element* $\mathbf{g} \in \mathcal{G}(S)$ *belongs to this subgroup* $\mathcal{G}_0(S)$ *if and only if the associated roots of unity* $\{\rho_s\}$ *of equation* (3.1) *satisfy*

$$\rho_s \;=\; \begin{cases} \pm 1 & \text{if } s \text{ is a free vertex with degree } d(s) \text{ equal to } 2\,, \text{ but} \\ +1 & \text{in all other cases}\,. \end{cases}$$

Thus $\mathcal{G}_0(S)$ *can be described as a direct sum of copies of the group* $\{\pm 1\}$*, with one copy for each free vertex of degree two.*

Proof. First note that $\rho_{F(s)} = 1$ whenever $d(s) \ge 2$: This follows by considering the constant term $a_0/\rho_{F(s)}$ in equation (3.3). Since every $s' \in F(|S|)$ is in the forward orbit of some vertex of degree $d(s) \ge 2$, it follows inductively from equation (3.2) that $\rho_{s'} = 1$ for all $s' \in F(|S|)$. Now let s be a free vertex. If $d(s) > 2$, then by considering the coefficient $\rho_s a_1/\rho_{F(s)}$ of the linear term in equation (3.3) we see that $\rho_s = 1$. But in the case $d(s) = 2$ this linear term does not appear, and we can conclude only that $\rho_s^{\,2} = 1$. This proves that any element of $\mathcal{G}$ which acts trivially must belong to $\mathcal{G}_0(S)$, and the converse statement follows similarly. $\square$

Remark 3.7. One can also consider the group $\mathrm{Aut}(S)$ consisting of all one-to-one maps $\phi : |S| \to |S|$ which commute with F and preserve the critical weight. This group acts faithfully on the space $|S|\times\mathbb{C}$, mapping each pair $(s,\, z)$ to $(\phi(s),\, z)$. The groups $\mathcal{G}(S)$ and $\mathrm{Aut}(S)$ together generate a split extension

$$1 \;\to\; \mathcal{G}(S) \;\to\; \widehat{\mathcal{G}}(S) \;\leftrightarrow\; \mathrm{Aut}(S) \to 1\,,$$

consisting of all compositions

$$\phi\circ\mathbf{g} \;:\; (s,\; z) \;\mapsto\; \big(\phi(s)\,,\; \rho_s z\big)\;.$$

This group acts faithfully on $|S|\times\mathbb{C}$, and hence acts (not always faithfully) on $\mathcal{P}^S$.

4. Blaschke Products and the Model Space $\mathcal{B}^S$.

This section will describe a topological model, based on Blaschke products, for hyperbolic components with mapping scheme S. First a review of some standard facts. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. For any $a \in \mathbb{D}$, there is one and only one Möbius transformation μ_a of the Riemann sphere which maps $\mathbb{D}$ onto itself satisfying

$$\mu_a(a) \;=\; 0\,, \qquad \text{and} \quad \mu_a(1) \;=\; 1\,.$$

It is given by

$$\mu_a(z) \;=\; k\,\frac{z-a}{1-\overline{a}\,z} \qquad \text{with} \qquad k = \frac{1-\overline{a}}{1-a}\,. \tag{4.1}$$

Lemma 4.1. *Any proper holomorphic map from $\mathbb{D}$ onto itself extends continuously over the closed disk $\overline{\mathbb{D}}$, and can be written uniquely as an d-fold* ***Blaschke product***

$$\beta(z) \;=\; \beta(1)\;\mu_{a_1}(z)\,\cdots\,\mu_{a_d}(z) \tag{4.2}$$

with $d \ge 1$, where $|\beta(1)| = 1$, and where $a_1\;\ldots,\, a_d$ are the (*not necessarily distinct*) *pre-images of zero.*

Remark 4.2. It follows that every such map β extends uniquely as a rational map of degree d from the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C}\cup\infty$ onto itself. It is not hard to check that this extended map commutes with the inversion $z \mapsto 1/\overline{z} \;=\; z/|z|^2$ in the unit circle. In particular, z is a critical point if and only if $1/\overline{z}$ is critical, and z is periodic if and only if $1/\overline{z}$ is periodic.

Proof of Lemma 4.1. Since β is a proper map, the absolute value $|\beta(z)|$ tends to one as $|z| \to 1$; and since β is onto, it has at least one zero $a_1 \in \mathbb{D}$. It follows that the quotient $\beta_1(z) = \beta(z)/\mu_{a_1}(z)$ is a well defined holomorphic function on $\mathbb{D}$. Furthermore $|\beta_1(z)| \to 1$ as $|z| \to 1$, and it follows from the maximum principle that $|\beta_1(z)| \le 1$ everywhere in $\mathbb{D}$. If this function $\beta_1(z)$ is constant, this completes the proof. But if $\beta_1(z)$ is non-constant, then it follows from the minimum principle that there is a zero $\beta_1(a_2) = 0$, and we can continue inductively, setting $\beta_2(z) = \beta_1(z)/\mu_{a_2}(z)$, and so on. This induction must stop after finitely many steps since a proper holomorphic map can have at most finitely many zeros, counted with multiplicity. □

LEMMA 4.3. *A proper holomorphic map β of degree $d \geq 2$ from the unit disk onto itself induces an d-to-one covering map from the circle $\partial\mathbb{D}$ onto itself. Such a map β has at most one fixed point in the open disk $\mathbb{D}$. If there is an interior fixed point, then there are exactly $d-1$ distinct boundary fixed points. On the other hand, if there is no interior fixed point, then all $d+1$ fixed points in the Riemann sphere, counted with multiplicity, must lie on the unit circle. In any case, there are $d-1$ critical points, counted with multiplicity, in the interior of $\mathbb{D}$, and none on the boundary.*

PROOF. If we set $z = e^{2\pi i\theta}$ and $\beta(z) = e^{2\pi i\phi(\theta)}$, then $d\phi/d\theta$ can be identified with the (modified) logarithmic derivative

$$\frac{d\log\beta(z)}{d\log(z)} = \frac{\beta'/\beta}{z'/z} = \frac{z\,\beta'}{\beta}$$

evaluated on the unit circle (where $\beta' = d\beta/dz$ and $z' = 1$). For $d = 1$, since the circle maps diffeomorphically onto itself, we have $d\phi/d\theta > 0$ everywhere, and the integral around the circle is given by $\oint d\phi = +1$. In the case $d > 1$, it follows from (4.2) that the logarithmic derivative is the sum of d such terms, hence we again have $d\phi/d\theta > 0$, but with $\oint d\phi = d$. Thus β induces an d-fold covering map from the unit circle onto itself. In particular, any equation of the form $\beta(z) = \text{constant} \in \partial\mathbb{D}$ has exactly d distinct solutions.

Now suppose that β has a fixed point in the open unit disk. Then after conjugating by a conformal automorphism, we may assume that the fixed point is $z = 0$. In that case, since $\mu_0(z) = z$, we can write

$$\beta(z) \;=\; \beta(1)\, z\, \mu_{a_2}(z)\cdots\mu_{a_d}(z) \qquad \text{with} \qquad |\beta(1)| = 1\,.$$

It follows immediately from this formula that $|\beta(z)| < |z|$ for all $z \neq 0$ in the open disk. Thus this fixed point is attracting, and is unique within $\mathbb{D}$. Furthermore, expressing the logarithmic derivative on the unit circle as an d-fold sum, as above, the first term is $+1$, hence $d\phi/d\theta = z\beta'(z)/\beta(z) > 1$ whenever $d > 1$.

A similar argument shows that there are exactly $d-1$ fixed points on the unit circle (all repelling). In fact the difference function $\theta \mapsto \phi - \theta$ is a covering map from the circle $\mathbb{R}/\mathbb{Z}$ onto itself with degree $d-1$, and the zeros of this difference in $\mathbb{R}/\mathbb{Z}$ are exactly the fixed points.

Finally, there can be no critical points on the boundary. In fact if the derivative vanishes at a boundary point z_0, then the map has local degree ≥ 2 at z_0, hence no neighborhood of z_0 within $\overline{\mathbb{D}}$ can map into $\overline{\mathbb{D}}$. □

REMARK 4.4. More precisely, whenever there is an interior fixed point, it is not difficult to show that the induced map on $\partial\mathbb{D}$ is topologically conjugate to the linear map $t \mapsto td$ of the circle $\mathbb{R}/\mathbb{Z}$. To prove this, construct a new homeomorphism

$$t \,:\, \partial\mathbb{D} \;\to\; \mathbb{R}/\mathbb{Z}$$

as follows. Choose one boundary fixed point z_0 and assign it the coordinate $t(z_0) = 0$. The d immediate pre-images of z_0 divide $\partial\mathbb{D}$ into d disjoint half-open arcs $A_0, A_1, \ldots, A_{d-1}$, numbered in counterclockwise order starting and ending at the point z_0. Note that each of these arcs maps bijectively onto the entire circle. Now define the function $t : \partial\mathbb{D} \to \mathbb{R}/\mathbb{Z}$ by the formula

$$t(z) \;=\; \sum_{k=0}^{\infty} a_k(z)/d^{k+1}\,,$$

where the integers $0 \le a_k(z) < d$ are defined by the condition $\beta^{\circ k}(z) \in A_{a_k(z)}$. Then evidently $a_k\big(\beta(z)\big) = a_{k+1}(z)$, and hence

$$t\big(\beta(z)\big) \;=\; \sum_{k=0}^{\infty} a_{k+1}(z)/d^{k+1} \;=\; t(z)\,d - a_0(z) \;\equiv\; t(z)\,d \pmod{\mathbb{Z}}\,.$$

Using the condition that $d\phi/d\theta$ is greater than some constant $c > 1$, it is not difficult to prove that the first k terms of this series determine the point z to an accuracy of $2\pi/c^k$. It then follows easily that the function $z \mapsto t(z) \in \mathbb{R}/\mathbb{Z}$ is a homeomorphism, as required.

In studying polynomial maps, we concentrated on those which are monic and centered. As a substitute for the monic condition, let us say that a Blaschke product β is **1-anchored** if $\beta(1) = 1$. However, we will need two different concepts of centering, depending on whether we are dealing with a periodic point or an aperiodic point of $|S|$.

Definition 4.5. Let $\beta : \mathbb{D} \to \mathbb{D}$ be a proper holomorphic map of degree $d \ge 1$. We will say that β is ***fixed point centered*** if $\beta(0) = 0$, and ***zeros centered*** if the sum

$$a_1 \,+\, \cdots \,+\, a_d$$

of the points in $\beta^{-1}(0)$ (counted with multiplicity) is equal to zero. (In the case $d = 1$, note that the only Blaschke product which is 1-anchored, and centered in either sense, is the identity map.)

In order to construct an appropriate topological model for hyperbolic components with a given reduced mapping scheme, we will need three lemmas.

Lemma 4.6. *Let β be a Blaschke product of degree $d \ge 2$ which has a fixed point z_0 in the open disk $\mathbb{D}$, and let h be a Möbius automorphism of the unit disk. Then the conjugate $\beta' = h^{-1} \circ \beta \circ h$ is fixed point centered if and only if $h(0) = z_0$, and is 1-anchored if and only if $h(1)$ is one of the $d-1$ fixed points of β on the boundary circle $\partial\mathbb{D}$. Thus, for each such β, there are $d-1$ possible choices for h.*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{h} & \mathbb{D} \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ \mathbb{D} & \xrightarrow{h} & \mathbb{D} \end{array} \tag{4.3}$$

The proof is immediate. □

Lemma 4.7. *Let β be an arbitrary Blaschke product of degree $d \ge 1$, and let h be a Möbius automorphism. Then the composition $\beta \circ h$ is 1-anchored if and only if $h(1)$ is one of the d points $z_1 \in \partial\mathbb{D}$ for which $\beta(z_1) = 1$. For each such z_1, there is a unique choice of h so that $\beta \circ h$ is zeros centered.*

$$\begin{tikzcd} \mathbb{D} \arrow[r, "h"] \arrow[dr, "\beta\circ h"'] & \mathbb{D} \arrow[d, "\beta"] \\ & \mathbb{D} \end{tikzcd} \tag{4.4}$$

The proof will depend on the following.

DEFINITION 4.8. Given points $z_1, \ldots, z_k$ in a Riemann surface W isomorphic to $\mathbb{D}$, it follows from Douady and Earle [**DE**, §2] that there exists a conformal isomorphism $\eta : W \to \mathbb{D}$, unique up to a rotation of $\mathbb{D}$, which takes the z_j to points with sum $\eta(z_1)+\cdots+\eta(z_k)$ equal to zero. By definition, the pre-image $\widehat{z} = \eta^{-1}(0) \in W$ is called the ***conformal barycenter*** of the points $z_1, \ldots, z_k \in W$. Evidently this conformal barycenter is uniquely defined.

PROOF OF LEMMA 4.7. If $h : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism, note that h maps the zeros of $\beta \circ h$ to those of β, and hence maps the conformal barycenter of the zeros of $\beta \circ h$ to the corresponding barycenter $\widehat{a}$ of the zeros $a_1, \ldots, a_n$ of β. In particular, it follows that $\beta \circ h$ is zeros centered if and only if $h(0) = \widehat{a}$. Using these facts, the proof is straightforward. □

The notations $\mathcal{B}^d_{\rm fc}$ and $\mathcal{B}^d_{\rm zc}$ will be used for the topological space consisting of all 1-anchored Blaschke products of degree d which are respectively fixed point centered or zeros centered. For the special case $d = 1$, evidently $\mathcal{B}^1_{\rm fc} = \mathcal{B}^1_{\rm zc}$ consists of a single point, namely the identity map.

LEMMA 4.9. *Each of the two model spaces* $\mathcal{B}^d_{\rm fc}$ *and* $\mathcal{B}^d_{\rm zc}$, *is homeomorphic to an open cell of real dimension* $2(d-1)$.

PROOF. Let $\mathcal{S}_d(\mathbb{C})$ be the d***-fold symmetric product***, consisting of unordered d-tuples $\{a_1, \ldots, a_d\}$ of complex numbers. This can be identified with the complex affine space consisting of all monic polynomials of degree d, under the correspondence

$$\{a_1, \ldots, a_d\} \mapsto (z-a_1)\cdots(z-a_d) = z^d - \sigma_1 z^{d-1} + \sigma_2 z^{d-2} - \cdots + (-1)^d \sigma_d\,,$$

where the σ_j are the elementary symmetric functions of $\{a_1, \ldots, a_d\}$. Thus $\mathcal{S}_d(\mathbb{C})$ is homeomorphic to $\mathbb{C}^d \cong \mathbb{R}^{2d}$. Since $\mathbb{C}$ is homeomorphic to the 2-cell $\mathbb{D}$, it follows that $\mathcal{S}_d(\mathbb{D})$ is also homeomorphic to $\mathbb{R}^{2d}$.

Now consider the space $\mathcal{B}^{\,d}_{\rm fc}$ consisting of 1-anchored Blaschke products β of degree d which fix the origin. We can write

$$\beta(z) \;=\; z\,\mu_{a_2}(z)\cdots\mu_{a_d}(z)$$

(taking $a_1 = 0$). Evidently this space is homeomorphic to the symmetric product $\mathcal{S}_{\mathbf{w}}(\mathbb{D})$ where $\mathbf{w} = d-1$, and hence is a topological cell, homeomorphic to $\mathbb{R}^{2\mathbf{w}}$.

To determine the topology of $\mathcal{B}^d_{\rm zc}$, we proceed as follows. We show first that the subspace $\mathcal{S}_d(\overline{\mathbb{D}}) \subset \mathcal{S}_d(\mathbb{C})$, consisting of unordered d-tuples $\{a_1, a_2, \ldots, a_d\}$ with $\max_j(|a_j|) \le 1$ is a closed topological $2d$-cell with interior equal to $\mathcal{S}_d(\mathbb{D})$. In fact, for each $\{a_1, \ldots, a_d\} \in \mathcal{S}_d(\mathbb{C})$ such that the maximum of the $|a_j|$ is equal to one, consider the half-line consisting of points $\{ta_1, \ldots, ta_d\}$ with $t \ge 0$. The image of each such half-line in the space of elementary symmetric n-tuples is a curve consisting of points $(t\sigma_1, t^2\sigma_2, \ldots, t^d\sigma_d) \in \mathbb{C}^d$, which crosses the unit sphere of

$\mathbb{C}^d$ exactly once, since the function $t \mapsto |t\,\sigma_1|^2 + \cdots + |t^d\,\sigma_d|^2$ is strictly monotone. Hence, stretching by an appropriate factor along each such ray, we obtain the required homeomorphism from $\mathcal{S}_d(\overline{\mathbb{D}})$ to the closed unit ball in $\mathbb{C}^d$. Using this construction we see also that the subspace of $\mathcal{S}_d(\mathbb{D})$ consisting of unordered d-tuples with sum $\sigma_1 = a_1 + \cdots + a_d$ equal to zero is an open topological $2(d-1)$-cell. Thus the set $\mathcal{B}^d_{\rm zc}$ consisting of Blaschke products of the form $\beta(z) = \mu_{a_1}(z)\cdots\mu_{a_d}(z)$ with $a_1 + \cdots + a_d = 0$ is an open topological $2(d-1)$-cell. $\square$

Combining the three previous lemmas, we can construct an appropriate topological model for hyperbolic components with a given reduced mapping scheme.

Definition 4.10. To any mapping scheme $S = (|S|\,,\,F\,,\,\mathbf{w})$ we associate the ***model space*** $\mathcal{B}^S$ consisting of all proper holomorphic maps

$$\boldsymbol{\beta} : |S| \times \mathbb{D} \;\to\; |S| \times \mathbb{D}$$

such that $\boldsymbol{\beta}$ carries each $s \times \mathbb{D}$ onto $F(s) \times \mathbb{D}$ by a 1-anchored Blaschke product

$$(s,\, z) \;\mapsto \big(F(s)\,,\; \beta_s(z)\big)$$

of degree $d(s) = \mathbf{w}(s) + 1$ which is either fixed point centered or zeros centered according as s is periodic or aperiodic under F. (Thus, in the special case of a vertex s of weight zero, we require β_s to be the identity map.)

Lemma 4.11. *If the scheme S has total weight $\mathbf{w}(S)$, then the model space $\mathcal{B}^S$ is homeomorphic to an open cell of dimension $2\mathbf{w}(S)$. Furthermore, $\mathcal{B}^S$ is canonically homeomorphic to $\mathcal{B}^{\overline{S}}$, where $\overline{S}$ is the associated reduced mapping scheme.*

Proof. The first statement follows immediately from Lemma 4.9 since, as a topological space, $\mathcal{B}^S$ is simply a Cartesian product of spaces of the form $\mathcal{B}^{d(s)}_{\rm fc}$ and $\mathcal{B}^{d(s)}_{\rm zc}$ of dimension $2\mathbf{w}(s) = 2(d(s)-1)$, where $\sum \mathbf{w}(s) = \mathbf{w}(S)$. The second statement follows since the space $\mathcal{B}^1_{\rm fp} = \mathcal{B}^1_{\rm zc}$ is a single point. $\square$

We will show that the various maps in $\mathcal{B}^S$ serve as models for the dynamics of all possible hyperbolic components H with $S_H \cong S$. As a preliminary step, given a mapping scheme S, first consider more general maps

$$\boldsymbol{\beta} : |S| \times \mathbb{D} \to |S| \times \mathbb{D}$$

which carry each $s \times \mathbb{D}$ onto $F(s) \times \mathbb{D}$ by a Blaschke product of degree $d(s)$, but with no other restriction. Evidently, each such $\boldsymbol{\beta}$ extends uniquely over the union $|S| \times \overline{\mathbb{D}}$ of closed disks.

Definition 4.12. By a ***boundary marking*** q for $\boldsymbol{\beta}$ we will mean a function $s \mapsto q(s) \;\in\; s \times \partial\mathbb{D}$ which assigns a boundary point to each $s \times \overline{\mathbb{D}}$, and which satisfies the condition that

$$q\big(F(s)\big) \;=\; \boldsymbol{\beta}\big(q(s)\big) \qquad \text{for every} \quad s\,. \tag{4.5}$$

Lemma 4.13. *Given $\boldsymbol{\beta}$ as above, the number of possible boundary markings q is equal to the order of the automorphism group $\mathcal{G}(S)$, as computed in Lemma 3.3. In particular, such boundary markings always exist.*

PROOF. First consider a vertex s which is periodic, $F^{\circ k}(s) = s$. If $d_1 d_2 \cdots d_k$ is the product of the degrees around this periodic orbit, then we can choose $q(s)$ to be any one of the $d_1 \cdots d_k - 1$ fixed points of $\boldsymbol{\beta}^{\circ k}$ on $s \times \partial\mathbb{D}$. (Compare Lemma 4.3.) The choice of $q\big(F(s)\big)$ is then determined by equation (4.5), and we can continue inductively around the cycle. In the case of an aperiodic vertex, suppose inductively that $q\big(F(s)\big)$ has already been chosen, but $q(s)$ has not. Then there are $d(s)$ possible choices for $q(s)$, again by Lemma 4.3. Further details of the proof are straightforward. □

THEOREM 4.14. *As above, let* $\boldsymbol{\beta} : |S| \times \mathbb{D} \to |S| \times \mathbb{D}$ *carry each* $s \times \mathbb{D}$ *onto* $F(s) \times \mathbb{D}$ *by a proper holomorphic map of degree* $d(s)$. *Suppose that* $\boldsymbol{\beta}$ *is* ***hyperbolic*** *in the sense that every orbit in* $|S| \times \mathbb{D}$ *converges to an attracting cycle in* $|S| \times \mathbb{D}$. *Then for every boundary marking* q *there exists a unique automorphism* $\mathbf{h}$ *of* $|S| \times \overline{\mathbb{D}}$ *such that the conjugate map* $\mathbf{h}^{-1} \circ \boldsymbol{\beta} \circ \mathbf{h}$ *belongs to the space* $\mathcal{B}^S$ *of Definition* 4.10, *with* $\mathbf{h}(s, 1) = q(s)$.

PROOF. If we choose $\mathbf{h}$ so that $\mathbf{h}(s, 1) = q(s)$ for every s, then it is straightforward to check that $\mathbf{h}^{-1} \circ \boldsymbol{\beta} \circ \mathbf{h}$ is 1-anchored. We must show that there is then a unique choice of the values $\mathbf{h}(s,\, 0)$ so that $\mathbf{h}^{-1} \circ \boldsymbol{\beta} \circ \mathbf{h}$ also satisfies the appropriate centering conditions. If s is periodic under F, then since $\boldsymbol{\beta}$ is hyperbolic, it follows that $s \times \mathbb{D}$ contains a necessarily unique attracting periodic point $(s,\, z_s)$ and we can choose $\mathbf{h}$ so that $\mathbf{h}(s, 0) = (s,\, z_s)$. On the other hand, if s is not periodic, then by Lemma 4.7, assuming inductively that $\mathbf{h}$ has already been defined on $F(s) \times \mathbb{D}$, we can choose the automorphism $\mathbf{h}$ on $s \times \mathbb{D}$ so that $\mathbf{h}^{-1} \circ \boldsymbol{\beta} \circ \mathbf{h}$ is zeros centered on $s \times \mathbb{D}$. The rest of the argument is straightforward. □

DEFINITION 4.15. If $\mathbf{f}$ is a hyperbolic map in the connectedness locus of the space $\mathcal{P}^{S_0}$ for some scheme S_0, let $\mathcal{U}_{\mathbf{f}} \subset |S_0| \times \mathbb{C}$ be the union of those Fatou components of $\mathbf{f}$ which contain critical or postcritical points.

COROLLARY 4.16. *Let* $\mathbf{f}$ *and* $\mathcal{U}_{\mathbf{f}}$ *be as above. Then the map* $\mathbf{f}$ *restricted to* $\mathcal{U}_{\mathbf{f}}$ *is conformally conjugate to some map* $\boldsymbol{\beta} : |S_{\mathbf{f}}| \times \mathbb{D} \to |S_{\mathbf{f}}| \times \mathbb{D}$ *belonging to the model space* $\mathcal{B}^{S_{\mathbf{f}}}$. *A similar assertion holds for rational maps which are hyperbolic, with connected Julia set.*

PROOF. Since each connected component of $\mathcal{U}_{\mathbf{f}}$ is conformally isomorphic to the unit disk, we can choose a conformal isomorphism from $\mathcal{U}_{\mathbf{f}}$ to $|S_{\mathbf{f}}| \times \mathbb{D}$. The conclusion then follows easily from Lemma 4.13 and Theorem 4.14. □

We will also need the following elementary result.

LEMMA 4.17. *Each model space* $\mathcal{B}^S$ *contains one and only one map* $\boldsymbol{\beta}_0$ *which is critically finite, given by the formula*

$$\boldsymbol{\beta}_0(s, z) \;=\; \big(F(s),\, z^{d(s)}\big)\,.$$

PROOF. First consider a Blaschke product $\beta : \mathbb{D} \to \mathbb{D}$ of degree $n \geq 2$, which is critically finite. Then β certainly has a periodic point, say of period k. If $k > 1$ then the k-fold iterate of β would have k distinct fixed points, which contradicts Lemma 4.3; therefore, $k = 1$. After conjugating by a Möbius automorphism, we may assume that this fixed point lies at the origin. We must then prove that the origin is the only critical point. If $\beta'(0)$ were non-zero, then we could choose a Kœnigs coordinate, defined throughout a maximal open set $U \subset \mathbb{D}$ which maps diffeomorphically onto a round disk. The boundary ∂U would then have to contain a critical point with infinite orbit, which contradicts the hypothesis. On the other hand, if $\beta'(0) = 0$, then one can choose a Böttcher coordinate mapping a maximal open set U diffeomorphically onto a round disk. If $U \neq \mathbb{D}$, then ∂U would again contain a critical point with infinite orbit, contradicting the hypothesis. Thus the origin is the only critical point. After conjugating by a rotation, it follows that $\beta(w) = w^n$.

Now consider a critically finite $\boldsymbol{\beta} \in \mathcal{B}^S$. If $s \in |S|$ is a periodic vertex of period k, then the argument above shows that $\boldsymbol{\beta}^{\circ k}(s,\, w) = (s,\, w^n)$, where n is the degree of $\boldsymbol{\beta}^{\circ k}$ on $s \times \mathbb{D}$. It follows that $\boldsymbol{\beta}$ has no critical points on $s \times \mathbb{D}$ other than $(s,\, 0)$; therefore $\boldsymbol{\beta}(s,\, w) = (s,\, w^{d(s)})$. Now consider an aperiodic vertex s. Assuming inductively that the only periodic or preperiodic point in $F(s) \times \mathbb{D}$ is $\big(F(s),\, 0\big)$, it follows that $\boldsymbol{\beta}$ must map every critical point in $s \times \mathbb{D}$ to $\big(F(s),\, 0\big)$. Let $s \times X$ be the finite set consisting of all pre-images of $\big(F(s),\, 0\big)$ in $s \times \mathbb{D}$. Then $s \times (\mathbb{D} \smallsetminus X)$ is an unbranched $d(s)$-fold covering space of $F(s) \times (\mathbb{D} \smallsetminus \{0\})$. Therefore $\mathbb{D} \smallsetminus X$ is conformally isomorphic to a punctured disk. This implies that X is a single point, which must be the origin since this map is zeros centered. This completes the induction, and hence completes the proof. $\square$

5. Hyperbolic Components are Topological Cells.

Given any mapping scheme S_0, consider the associated space $\mathcal{P}^{S_0}$ of polynomial maps (Definition 2.2). Let $H \subset \mathcal{P}^{S_0}$ be a hyperbolic component in the connectedness locus of $\mathcal{P}^{S_0}$.

As in Definition 4.15 for each $\mathbf{f} \in H$, let $\mathcal{U}_\mathbf{f} \subset K(\mathbf{f})$ be the union of those Fatou components of $\mathbf{f}$ which contain critical or postcritical points.

The object of this section is to prove the following result.

Theorem 5.1. *Let S be the full mapping scheme associated with some representative map in H. Then there exists a diffeomorphism $H \xrightarrow{\cong} \mathcal{B}^S$ which sends each $\mathbf{f} \in H$ to a map $\boldsymbol{\beta}(f) : |S| \times \mathbb{D} \to |S| \times \mathbb{D}$ in $\mathcal{B}^S$ which is conformally conjugate to the restriction $\mathbf{f}|_{\mathcal{U}_\mathbf{f}} : \mathcal{U}_\mathbf{f} \to \mathcal{U}_\mathbf{f}$.*

In fact the proof will show that there exist only finitely many such diffeomorphisms, where the number of possible choices is equal to the order of the automorphism group $\mathcal{G}(S)/\mathcal{G}_0(S)$.

Combining this statement with Lemmas 4.11 and 4.17, we immediately obtain the following result, which generalizes an unpublished theorem of McMullen.

Corollary 5.2. *Every such hyperbolic component H is a topological cell of dimension $2\,\mathbf{w}(S)$; and every such H contains a unique critically finite map.*

We must be careful in the proof of Theorem 5.1 since it is not a priori clear that H is simply connected. As an example to illustrate the difficulty, each $\mathbf{f} \in H$ has a mapping scheme $S_\mathbf{f}$. Following a path from $\mathbf{f}$ to $\mathbf{f}'$, we obtain a well defined isomorphism from $S_\mathbf{f}$ to $S_{\mathbf{f}'}$. However, we must check that this isomorphism does not depend on the choice of path.

In analogy with Definition 4.12, we introduce the concept of boundary markings for hyperbolic maps. Recall that the boundary of each component of $\mathcal{U}_{\mathbf{f}}$ is a Jordan curve.[2]

DEFINITION 5.3. A ***boundary marking for a hyperbolic map*** $\mathbf{f} \in \mathcal{C}^{S_0}$ will mean a function q which assigns to each connected component $U \subset \mathcal{U}_{\mathbf{f}}$ a boundary point $q(U) \in \partial U$ so as to satisfy the identity

$$q\big(\mathbf{f}(U)\big) = \mathbf{f}\big(q(U)\big)\,.$$

DEFINITION 5.4. Let S be the full mapping scheme for some representative map in H, and let $\widetilde{H}$ be the set of all triples consisting of

- a map $\mathbf{f} \in H$,
- a boundary marking q for $\mathbf{f}$, and
- an isomorphism $\boldsymbol{\iota} : S_{\mathbf{f}} \;\cong\; S$.

LEMMA 5.5. *This set $\widetilde{H}$ has a natural topology so that every point of H has a neighborhood N which is evenly covered*[3] *under the projection $\widetilde{H} \to H$.*

PROOF. This is straightforward. In fact, each point $q(U)$ is preperiodic and eventually repelling, and therefore deforms continuously as we deform the map $\mathbf{f}$. Similarly the isomorphism $\boldsymbol{\iota}$ deforms continuously with $\mathbf{f}$. $\square$

It follows that every connected component of $\widetilde{H}$ is a (possibly trivial) covering space of H. (It also follows that we can lift the complex structure from H, so that the projection map is locally biholomorphic.)

Next we project this space $\widetilde{H}$ onto the model space $\mathcal{B}^S$.

LEMMA 5.6. *To every $(\mathbf{f},\, q,\, \boldsymbol{\iota}) \in \widetilde{H}$, there is uniquely associated a map*

$$\boldsymbol{\beta} = \pi(\mathbf{f},\, q,\, \boldsymbol{\iota}) \in \mathcal{B}^S\,,$$

together with a conformal conjugacy between the restriction $\mathbf{f}|_{\mathcal{U}_{\mathbf{f}}} : \mathcal{U}_{\mathbf{f}} \to \mathcal{U}_{\mathbf{f}}$ and the map $\boldsymbol{\beta} : |S| \times \mathbb{D} \to |S| \times \mathbb{D}$.

PROOF. We use the isomorphism $\boldsymbol{\iota}$ to identify $S_{\mathbf{f}}$ with S. Start with some arbitrary conformal isomorphism which carries each component $U \subset \mathcal{U}_{\mathbf{f}}$ onto the corresponding $s_U \times \mathbb{D}$. Then the boundary marking $q(U) \in \partial U$ will correspond to a boundary marking in $s_U \times \partial\mathbb{D}$. We can then use Theorem 4.14 to obtain a corrected conformal isomorphism $\mathcal{U}_{\mathbf{f}} \to |S| \times \mathbb{D}$ which is actually a conformal conjugacy between $\mathbf{f}|_{\mathcal{U}_{\mathbf{f}}}$ and a corresponding map $\boldsymbol{\beta} = \pi(\mathbf{f},\, \boldsymbol{\iota},\, q) \in \mathcal{B}^S$. $\square$

THEOREM 5.7. *The resulting projection $\pi : \widetilde{H} \to \mathcal{B}^S$ is continuous. Furthermore, every point $\boldsymbol{\beta} \in \mathcal{B}^S$ has a neighborhood N which is evenly covered.*

---

[2]PROOF. (For a more general result, see [**RY**].) It suffices to consider the classical case of a polynomial map $f : \mathbb{C} \to \mathbb{C}$. Since f is hyperbolic with connected Julia set, its Julia set is locally connected. Therefore, for any bounded Fatou component U, a conformal equivalence $\mathbb{D} \xrightarrow{\cong} U$, extends to a continuous map $\overline{\mathbb{D}} \to \overline{U}$. If two points $e^{i\theta}$ and $e^{i\theta'}$ in $\partial\mathbb{D}$ mapped to the same point of $z \in \partial U$, then the broken line from $e^{i\theta}$ to 0 to $e^{i\theta'}$ would map to a simple closed curve $\Gamma \subset \mathbb{C}$. By the maximum modulus principle, the bounded component of the complement of Γ must lie in the interior of the filled Julia set, and hence must be contained in U. But this would imply that there is an entire interval of angles, say with $\theta \le \phi \le \theta'$, so that $e^{i\phi}$ maps to z. This is impossible by a theorem of Riesz and Riesz. (See for example [**M4**, §§17.14, 19.2 and A.3].)

[3]By definition, N is ***evenly covered*** if each connected component of $p^{-1}(N)$ maps homeomorphically onto N.

Remark 5.8. If we assume this theorem for the moment, then the main results of this section follow easily. Since $\mathcal{B}^S$ is a topological cell by Lemma 4.11, it is certainly simply-connected. Thus Theorem 5.7 implies that every connected component of $\widetilde{H}$ maps homeomorphically onto $\mathcal{B}^S$. Since $\mathcal{B}^S$ has a unique critically finite point by Lemma 4.17, it follows that each connected component of $\widetilde{H}$ also has a unique critically finite point. On the other hand, each connected component of $\widetilde{H}$ is a covering space of H by Lemma 5.5. In fact it must actually map homeomorphically onto H; for if it were a non-trivial covering space, then $\widetilde{H}$ would have more than one critically finite point.

Thus, choosing any section $H \to \widetilde{H}$, the composition

$$H \longrightarrow \widetilde{H} \xrightarrow{\pi} \mathcal{B}^S$$

maps H homeomorphically onto the topological cell $\mathcal{B}^S$. This shows that Theorem 5.1 and Corollary 5.2, as stated at the beginning of this section, follow immediately from Theorem 5.7.

The proof of Theorem 5.7 will make use of the following.

Lemma 5.9. *Given $\boldsymbol{\beta} \in \mathcal{B}^S$, it is possible to choose a radius $0 < r(s) < 1$ for each $s \in |S|$ satisfying the following two conditions:*

(1) *Every critical point of $\boldsymbol{\beta}$ in $s \times \mathbb{D}$ is contained in $s \times \mathbb{D}_{r(s)}$, where $\mathbb{D}_r \;=\; \{w \in \mathbb{C}\;;\; |w| < r\}$ denotes the open disk of radius r.*
(2) *The image of the closure of this disk under $\boldsymbol{\beta}$ satisfies*

$$\boldsymbol{\beta}\big(s \times \overline{\mathbb{D}}_{r(s)}\big) \;\subset\; F(s) \times \mathbb{D}_{r(F(s))}\,. \tag{5.1}$$

Proof. Start with the aperiodic vertices. If s belongs to the set $S' = S \smallsetminus F(S)$ of "free" vertices, then any $r(s)$ sufficiently close to one will do. Next choose $r(s)$ for the aperiodic vertices in $F(S')$, and continue inductively. Once $r(s)$ has been chosen for all aperiodic vertices, the remaining choices are not difficult. In fact all of the maps around a cycle are fixed point centered. Therefore, for s periodic,

$$\text{if} \qquad \boldsymbol{\beta}(s,\, w) = \big(F(s),\, w'\big)\,, \qquad \text{then} \qquad |w'| \;\le\; |w|\,,$$

with strict inequality whenever $w \neq 0$ and $d(s) \geq 2$ by the Schwarz Lemma. Further details are straightforward, since every cycle contains at least one vertex of degree ≥ 2. □

It is now easy to choose radii $R(s)$ slightly larger than $r(s)$ so that

$$\boldsymbol{\beta}\big(s \times \overline{\mathbb{D}}_{R(s)}\big) \;\subset\; F(s) \times \mathbb{D}_{r(F(s))}\,. \tag{5.2}$$

The annuli

$$A(s) \;=\; s \times \big(\mathbb{D}_{R(s)} \smallsetminus \overline{\mathbb{D}}_{r(s)}\big)$$

will play an important role. Note the crucial property that no orbit under $\boldsymbol{\beta}$ can pass through the union $\bigcup_s A(s)$ more than once.

Proof of Theorem 5.7. Recall that $H \subset \mathcal{P}^{S_0}$ is a hyperbolic component with mapping scheme S; that $\widetilde{H}$ is the finite covering space of H consisting of triples $(\mathbf{f},\, q,\, \boldsymbol{\iota})$; and that

$$\pi : \widetilde{H} \to \mathcal{B}^S$$

is the associated projection map. Given any $\boldsymbol{\beta} \in \mathcal{B}^S$, we must find a neighborhood N of $\boldsymbol{\beta}$ which is evenly covered. This means that, given any $(\mathbf{f}_0,\, q_0,\, \boldsymbol{\iota}_0) \in \pi^{-1}(N)$,

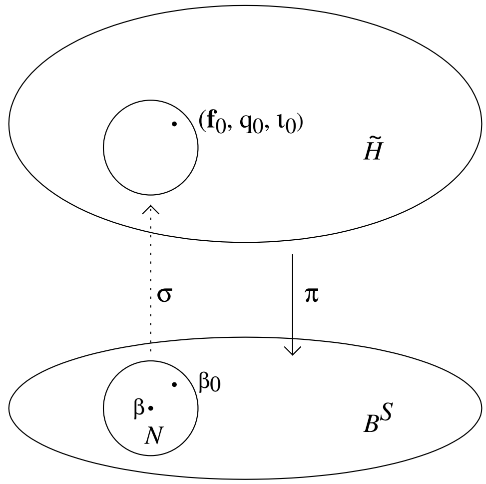


FIGURE 3. *Proof of the even covering property.*

and setting $\boldsymbol{\beta}_0 = \pi(\mathbf{f}_0, q_0, \boldsymbol{\iota}_0) \in N$, we must find a section $\sigma : N \to \widetilde{H}$ such that $\sigma(\boldsymbol{\beta}_0) = (\mathbf{f}_0, q_0, \boldsymbol{\iota}_0)$, with $\pi \circ \sigma$ equal to the identity map of N. To achieve this, we will impose two conditions on N:

CONDITION 1. *This neighborhood N must be small enough so that the conditions* (1) *and* (2) *of Lemma* 5.9*, as well as inequality* (5.2)*, hold with the same choice of radii $r(s)$, $R(s)$ for all $\boldsymbol{\beta}' \in N$.*

Let $\mathcal{U}_{\mathbf{f}_0}$ be the union of all critical and postcritical Fatou components for $\mathbf{f}_0$. Using Lemma 5.6, we can identify $|S| \times \mathbb{D}$ with this open set $\mathcal{U}_{\mathbf{f}_0} \subset |S_0| \times \mathbb{C}$. Furthermore, under this identification, the map $\boldsymbol{\beta}_0$ from $|S| \times \mathbb{D}$ to itself corresponds to the map $\mathbf{f}_0$ from $\mathcal{U}_{\mathbf{f}_0}$ to itself.

For any $\boldsymbol{\beta}_1 \in N$, the image $\sigma(\boldsymbol{\beta}_1) \in \pi^{-1}(\boldsymbol{\beta}_1) \subset \widetilde{H}$ will be constructed by quasiconformal surgery. (Compare [**DH2**].) The first step is to construct a preliminary map $\widehat{\mathbf{f}}_1$ from $|S_0| \times \mathbb{C}$ to itself as follows. Set $\widehat{\mathbf{f}}_1(s, w) = \boldsymbol{\beta}_1(s, w)$ whenever (s, w) belongs to the small disk

$$s \times \mathbb{D}_{r(s)} \ \subset \ |S| \times \mathbb{D} \ \cong \ \mathcal{U}_{\mathbf{f}_0} \ \subset \ |S_0| \times \mathbb{C}\,.$$

On the other hand, let $\widehat{\mathbf{f}}_1$ coincide with $\boldsymbol{\beta}_0$ outside the union of larger disks

$$\bigcup_s s \times \mathbb{D}_{R(s)} \ \subset \ |S| \times \mathbb{D} \ \cong \ \mathcal{U}_{\mathbf{f}_0} \ \subset \ |S_0| \times \mathbb{C}\,.$$

Within the intermediate closed annuli $\overline{A}(s) = s \times \left(\overline{\mathbb{D}}_{R(s)} \smallsetminus \mathbb{D}_{r(s)}\right)$, we interpolate linearly, setting

$$\widehat{\mathbf{f}}_1(s, w) \ = \ t\,\boldsymbol{\beta}_0(s, w) \,+\, (1-t)\,\boldsymbol{\beta}_1(s, w)\,, \qquad \text{where} \qquad t \ = \ \frac{|w| - r(s)}{R(s) - r(s)}\,.$$

We can now impose the second condition:

Condition 2. *The neighborhood N must be small enough so that, for each $\boldsymbol{\beta}_1$ in N, the map $\widehat{\mathbf{f}}_1$ defined in this way has Jacobian determinant bounded away from zero throughout each $\overline{A}(s)$.*

Next we will use quasiconformal surgery to construct a new conformal structure on $S_0 \times \mathbb{C}$ which is $\widehat{\mathbf{f}}_1$ invariant. To do this, start with the standard (quasi-) conformal structure on the small disks $s \times \mathbb{D}_{r(s)}$, and also on all points of $|S_0| \times \mathbb{C}$ which are not in the iterated pre-image of $\mathcal{U}_{\mathbf{f}_0}$. Now pull this quasiconformal structure back to the rest of $\mathcal{U}_{\mathbf{f}_0}$ under the action of $\widehat{\mathbf{f}}_1$ and its iterates. This will yield a well defined quasiconformal structure on $|S_0| \times \mathbb{C}$, which has bounded dilatation since an orbit can pass through the union of annuli $A(s)$ at most once. Using the measurable Riemann mapping theorem, we can choose a straightening map η which carries each $s \times \mathbb{C}$ to itself, and which carries our exotic quasiconformal structure to the standard conformal structure. This implies that the map $\mathbf{f}_1 = \eta^{-1} \circ \widehat{\mathbf{f}}_1 \circ \eta$ is holomorphic with respect to the standard conformal structure. Now, after composing η with suitable component-wise affine transformations, we may assume that $\mathbf{f}_1$ is 1-anchored and centered. Using the Ahlfors-Bers measurable Riemann mapping theorem with parameters [**AB**], we can choose these affine transformations so that $\mathbf{f}_1$ varies continuously as $\boldsymbol{\beta}_1$ varies over the neighborhood N. Now we define the section $\sigma : N \to \widetilde{H}$ by setting $\sigma(\boldsymbol{\beta}_1) = (\mathbf{f}_1,\, q_1,\, \boldsymbol{\iota}_1)$ where $\boldsymbol{\iota}_1$ is constant and where the boundary marking q_1 also varies continuously with $\boldsymbol{\beta}_1$.

Next, we must prove that $\pi(\mathbf{f}_1,\, q_1,\, \boldsymbol{\iota}_1) = \boldsymbol{\beta}_1$. The proof will depend on the following.

Lemma 5.10. *The conformal conjugacy class of a map $\boldsymbol{\beta}_1 \in N$ is uniquely determined by the conformal conjugacy class of the restriction of $\boldsymbol{\beta}_1$ to the union $\bigcup_s s \times \mathbb{D}_{r(s)}$ of subdisks.*

Proof. Define sets $\mathbb{D}_k(s) \subset s \times \mathbb{D}$ inductively by setting $\mathbb{D}_0(s) = s \times \mathbb{D}_{r(s)}$ and

$$\mathbb{D}_{k+1}(s) \;=\; (s \times \mathbb{D}) \;\cap\; \boldsymbol{\beta}_1^{-1}\mathbb{D}_k\big(F(s)\big)\ .$$

Then it is not hard to check that each $\mathbb{D}_k(s)$ is an open topological disk with smooth boundary, and that $\mathbb{D}_0(s) \subset \mathbb{D}_1(s) \subset \cdots$, with union $s \times \mathbb{D}$. Because all the critical values are well inside the disks, each $\mathbb{D}_{k+1}(s)$ can be described conformally as a $d(s)$-fold branched covering of $\mathbb{D}_k\big(F(s)\big)$, where the nontrivial branching already occurs in the subset $\mathbb{D}_0(s)$. Thus we can build these sets up inductively, starting only with the restriction of $\boldsymbol{\beta}_1$ mapping $\bigcup_s \mathbb{D}_0(s)$ into itself. The union is the required conformal dynamical system, conformally conjugate to $\boldsymbol{\beta}_1 : |S| \times \mathbb{D} \to |S| \times \mathbb{D}$. $\square$

Proof of Theorem 5.7, conclusion. Applying Lemma 5.10 to the map $\mathbf{f}_1$ as constructed above, it follows that $\pi(\mathbf{f}_1,\, q_1,\, \boldsymbol{\iota}_1) \in \mathcal{B}^S$ is conformally conjugate to $\boldsymbol{\beta}_1$. Since the boundary marking varies continuously with $\boldsymbol{\beta}_1$, this implies that $\pi(\mathbf{f}_1,\, q_1,\, \boldsymbol{\iota}_1) = \boldsymbol{\beta}_1$ as required.

Finally, we must prove that the projection $\pi : \widetilde{H} \to \mathcal{B}^S$ is also continuous. But the map $\sigma : N \to \sigma(N) \subset \widetilde{H}$ constructed above is known to be continuous and one-to-one; hence it maps any compact subset of N homeomorphically. Since the projection π can be identified locally with σ^{-1}, it follows that π is continuous. This completes the proof of Theorem 5.7, and hence of Theorem 5.1 and Corollary 5.2. $\square$

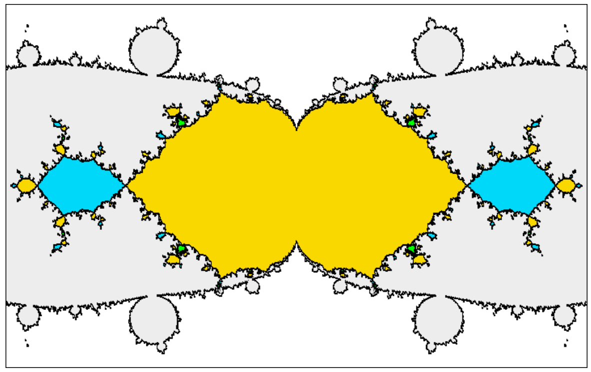


FIGURE 4. *The b parameter plane for the family of cubic maps $z \mapsto z^3 - 1.5\,z + b$. The large central region corresponds to a hyperbolic component with full mapping scheme of the form* $\bullet \leftrightarrow \bullet$ *of bitransitive type, while the two side regions have schemes of the form* $\bullet \to \cdot \to \bullet \leftrightarrow \cdot$ *of capture type.* (*At either of the two common boundary points $b = \pm 0.4202\cdots$, the attracting period two orbit persists, but one critical point becomes preperiodic.*) *In the surrounding light grey region, one critical orbit escapes to infinity, while in the outer white region both critical orbits escape.*

REMARK 5.11. In degrees ≥ 3, one cannot expect the boundaries of hyperbolic components to be smooth manifolds. (See Figure 4, and compare [**PTL**].) In fact, it is not at all clear that the boundary of every hyperbolic component must be a topological sphere. (Compare the discussion in Example 9.8.)

REMARK 5.12. It is often useful to restrict attention to some holomorphic subvariety of the parameter space $\mathcal{P}^{S_0}$. In general, one can't expect hyperbolic components in such a parameter subspace to satisfy Corollary 5.2. For example, the parameter slice shown in Figure 4 contains no critically finite points. However, in the special case where the subspace is defined by requiring one or more critical points to be periodic of specified period, the proof can be adapted as follows. Let $H \subset \mathcal{P}^{S_0}$ be a hyperbolic component with mapping scheme S. For each periodic vertex s of S, write the weight $\mathbf{w}(s)$ as a sum $\mathbf{w}'(s) + \mathbf{w}''(s)$ where $\mathbf{w}'(s)$ is to be the number of free critical points in the corresponding Fatou component, and where the periodic point in this component is required to be a critical of point multiplicity at least $\mathbf{w}''(s)$ whenever $\mathbf{w}''(s) > 0$. In the Blaschke product model for H, this means that the maps $\boldsymbol{\beta}$ must have the form

$$\boldsymbol{\beta}(s,\, z) \;=\; \big(F(s)\,,\; z^{\mathbf{w}''(s)+1}\mu_{a_1}(z)\cdots\mu_{a_{\mathbf{w}'(s)}}(z)\big)\;.$$

It is straightforward to check that the subspace of $\mathcal{B}_{\rm fc}^{\mathbf{w}(s)}$ defined in this way is a topological cell of dimension $2\mathbf{w}'(s)$ with a preferred center point. Since the full model space $\mathcal{B}^S$ is homeomorphic to a cartesian product of cells $\mathcal{B}_{\rm fc}^{\mathbf{w}(s)}$ with s periodic, together with cells $\mathcal{B}_{zc}^{\mathbf{w}(s')}$ with s' aperiodic, it follows that the subspace of $\mathcal{B}^S$ defined by all of these conditions is a topological cell with dimension twice the number of free critical points. (Here all of the critical points associated with an aperiodic vertex are free by definition.) Furthermore, this cell contains a unique critically finite point. The analogous statements for the corresponding subspace of H then follow from Theorem 5.1.

Note however that this argument works only within the given hyperbolic component. If we try to form an analogous global subvariety of $\mathcal{P}^{S_0}$, we must first mark one or more critical points, which will usually change the global topology. (Compare Remark 9.10.)

6. Analytic Isomorphism between Hyperbolic Components.

If $H_\alpha \subset \mathcal{C}^{S_1}$ and $H_\beta \subset \mathcal{C}^{S_2}$ are two different hyperbolic components with reduced mapping scheme isomorphic to S, then by Theorem 5.1 there are diffeomorphisms

$$H_\alpha \overset{\cong}{\longrightarrow} \mathcal{B}^S \overset{\cong}{\longleftarrow} H_\beta\,,$$

uniquely defined up to a choice among finitely many boundary markings, or equivalently up to the action of the group $\mathcal{G}(S)/\mathcal{G}_0(S)$ on $\mathcal{B}^S$. The composition mapping H_α to H_β will be called a ***canonical diffeomorphism*** between these two sets. We will prove the following.

Theorem 6.1. *This canonical diffeomorphism $H_\alpha \to H_\beta$ between open subsets of complex affine spaces is biholomorphic.*

Definition 6.2. As ***standard model*** for hyperbolic components with scheme S we can take the hyperbolic component $H_0^S \subset \mathcal{P}^S$ which is centered at the map $\mathbf{f}_0(s,\, z) = \big(F(s),\, z^{d(s)}\big)$.

In particular, it follows from Theorem 6.1 that the canonical diffeomorphism from H_α to the standard model H_0^S is biholomorphic. Note that this diffeomorphism is unique up to the action of the finite group $\mathcal{G}/\mathcal{G}_0$ of linear automorphisms of H_0^S. The proof of Theorem 6.1 will be based on the following.

Definition 6.3. We will say that a map $\mathbf{f} \in \mathcal{P}^{S_1}$ satisfies a ***critical orbit relation*** if either

(1) the $\mathbf{w}$ critical points of $\mathbf{f}$ are not all distinct, or
(2) the associated critical orbits are not disjoint from each other, or
(3) some critical orbit is periodic or eventually periodic.

It is not difficult to show that the set of all $\mathbf{f}$ which satisfy some critical orbit relation forms a countable union of algebraic varieties in the affine space $\mathcal{P}^{S_1}$. However, we can make a sharper statement for the hyperbolic subset of $\mathcal{P}^S$.

Lemma 6.4. *Let Q_α be the subset consisting of maps in H_α which have no critical orbit relation. Then Q_α is a dense open subset of H_α.*

Proof. Given $\mathbf{f} \in H_\alpha$, as in Definition 4.15 let $\mathcal{U}_\mathbf{f}$ be the union of all Fatou components of $\mathbf{f}$ which contain critical or postcritical points. First consider the simplest case in which $\mathcal{U}_\mathbf{f}$ is connected. In other words, assume that all of the critical points of $\mathbf{f}$ lie in $\mathcal{U}_\mathbf{f}$, which must be the immediate attracting basin of an attracting fixed point $p_\mathbf{f}$. This means that the associated full mapping scheme S consists of a single vertex of weight $\mathbf{w}$. Note first, for any $\mathbf{f} \in Q_\alpha$, that the multiplier $\lambda_\mathbf{f}$ at $p_\mathbf{f}$ must be non-zero. For otherwise $p_\mathbf{f}$ would be a fixed critical point. Thus we can choose a Kœnigs linearizing function $\kappa_\mathbf{f} : \mathcal{U}_\mathbf{f} \to \mathbb{C}$ which maps a neighborhood of $p_{\mathbf{f}}$ biholomorphically onto a neighborhood of the origin, and satisfies

$$\kappa_{\mathbf{f}}\big(\mathbf{f}(z)\big) \;=\; \lambda_{\mathbf{f}}\, \kappa_{\mathbf{f}}(z)\,. \tag{6.1}$$

Let $c_{\mathbf{f}}^1, \ldots, c_{\mathbf{f}}^{\mathbf{w}}$ be the critical points of $\mathbf{f}$. (Here the superscripts are just labels; not exponents.) If $\mathbf{f}$ has no critical orbit relations, then these critical points must be distinct, and can be chosen as functions which vary holomorphically as $\mathbf{f}$ varies through a small neighborhood. Furthermore, the values $\kappa_{\mathbf{f}}(c_{\mathbf{f}}^j)$ must all be distinct and non-zero. The condition (6.1) determines the Kœnigs function $\kappa_{\mathbf{f}}$ only up to a multiplicative constant. It will be convenient to normalize this function so that

$$\kappa_{\mathbf{f}}(c_{\mathbf{f}}^1) \;=\; +1\,.$$

It is then not hard to show that $\kappa_{\mathbf{f}}(z)$ is holomorphic as a function of both $\mathbf{f}$ and z throughout this small neighborhood.

For any $\lambda \in \mathbb{D}\smallsetminus\{0\}$, let $\mathbb{T}_\lambda$ denote the compact torus which is obtained by identifying each $z \in \mathbb{C}\smallsetminus\{0\}$ with $\lambda\, z$, and hence with all multiples of the form $\lambda^k\, z$. Then it is not difficult to check that a map $\mathbf{f} \in H_\alpha$ has no critical orbit relations if and only if

(1) $\lambda_{\mathbf{f}} \;\neq\; 0$, and
(2) the images of the numbers $\kappa_{\mathbf{f}}(c_{\mathbf{f}}^j)$ under projection to $\mathbb{T}_{\lambda_{\mathbf{f}}}$ are all distinct.

Since these are both open and dense conditions, this proves Lemma 6.4 in the special case. The proof in the general case is completely analogous. Just choose one periodic point in each attracting cycle, and work with the associated multipliers, Kœnigs functions, and compact tori. Details are left to the reader. □

LEMMA 6.5. *For any $\mathbf{f}_1$ with no critical orbit relations the canonical diffeomorphism from H_α to H_β is biholomorphic throughout a neighborhood of $\mathbf{f}_1$.*

PROOF. Again, we first consider the special case where all of the critical points lie in the immediate attracting basin of a single attracting fixed point. With $\lambda_{\mathbf{f}}$ and the $\kappa_{\mathbf{f}}(c_f^j)$ as above, we will show that the mapping

$$\mathbf{f} \;\mapsto\; \big(\, \lambda_{\mathbf{f}}\,,\; \kappa_{\mathbf{f}}(c_{\mathbf{f}}^2)\,,\; \kappa_{\mathbf{f}}(c_{\mathbf{f}}^3))\,,\; \cdots\,,\; \kappa_{\mathbf{f}}(c_{\mathbf{f}}^{\mathbf{w}})\,\big) \;\in\; \mathbb{C}^{\mathbf{w}}\,, \tag{6.2}$$

constitutes a local holomorphic coordinate system as $\mathbf{f}$ varies over a small neighborhood of $\mathbf{f}_1$ in Q_α. (As above, we assume that the Kœnigs function has been normalized so that $\kappa_{\mathbf{f}}(c_{\mathbf{f}}^1) = 1$.) In fact given $\mathbf{f}_1$, and given these $\mathbf{w}$ coordinate values, we will show how to reconstruct the map $\mathbf{f} : \mathcal{U}_{\mathbf{f}} \to \mathcal{U}_{\mathbf{f}}$ up to conformal conjugacy.

Choose a sequence of connected open sets

$$N_{\mathbf{f}}(0) \;\subset\; N_{\mathbf{f}}(1) \;\subset\; N_{\mathbf{f}}(2) \;\subset\; \cdots \;\subset\; \mathcal{U}_{\mathbf{f}}$$

with union equal to the entire space $\mathcal{U}_{\mathbf{f}}$ as follows.

DEFINITION 6.6. Let $N_{\mathbf{f}}(\ell)$ be the connected component which contains the fixed point $p_{\mathbf{f}}$ in the open set

$$\{\; z \in \mathcal{U}_{\mathbf{f}}\;;\; |\kappa_{\mathbf{f}}(z)| < a/{\lambda_{\mathbf{f}}}^{\ell}\;\}\;.$$

Here the constant a should be small enough so that $\mathbf{f}$ has no critical points in the closure $\overline{N}_{\mathbf{f}}(1)$, and should be carefully chosen so that no critical orbit of $\mathbf{f}$ hits the boundary $\partial N_{\mathbf{f}}(1)$.

Here are three easily verified properties.

(a) If this condition is satisfied for some given map $\mathbf{f}_1$, then it will also be satisfied for any map $\mathbf{f}$ in a sufficiently small neighborhood of $\mathbf{f}_1$.

(b) Each such $\mathbf{f}$ maps $N_{\mathbf{f}}(1)$ biholomorphically onto the proper subset $N_{\mathbf{f}}(0)$ of itself.

(c) Each $N_{\mathbf{f}}(\ell+1)$ maps onto $N_{\mathbf{f}}(\ell)$ by a branched covering which is branched only over those critical values of $\mathbf{f}$ which lie in $N_{\mathbf{f}}(\ell)$. The topological pattern of this branching remains the same for all $\mathbf{f}$ close to $\mathbf{f}_1$.

(In this special case where $\mathcal{U}_{\mathbf{f}}$ is connected, the $N_{\mathbf{f}}(\ell)$ are all connected sets; but this will no longer be true in the general case considered later. But in all cases, each connected component of $N_{\mathbf{f}}(\ell)$ will be simply connected, with smooth boundary.)

Given $\mathbf{f}_1 \in Q_\alpha$, we must show that any $\mathbf{f}$ sufficiently close to $\mathbf{f}_1$ is uniquely determined by $\lambda_{\mathbf{f}}$, together with the numbers $\kappa_{\mathbf{f}}(c_{\mathbf{f}}^j)$. We will first show by induction on ℓ that the conformal conjugacy class of the restriction

$$\mathbf{f} : N_{\mathbf{f}}(\ell) \to N_{\mathbf{f}}(\ell) \tag{6.3}$$

is uniquely determined by this data. To begin the induction, we need only $\lambda_{\mathbf{f}}$ to determine the conformal conjugacy class of $\mathbf{f}$ restricted to $N_{\mathbf{f}}(1)$. Assuming that we have constructed $N_{\mathbf{f}}(\ell)$ and the restriction of $\mathbf{f}$ to this Riemann surface, we need only to know the precise branch points in order to construct a Riemann surface isomorphic to $N_{\mathbf{f}}(\ell+1)$ as a branched covering. But locally, for $\mathbf{f}$ near $\mathbf{f}_1$, each branch point $\mathbf{f}(c_{\mathbf{f}}^j)$ is uniquely determined by the Kœnigs coordinate $\kappa_{\mathbf{f}}(c_{\mathbf{f}}^j)$. The inclusion map of $N_{\mathbf{f}}(\ell)$ into $N_{\mathbf{f}}(\ell+1)$ is then determined inductively. In fact the required branched covering $N_{\mathbf{f}}(\ell+1) \to N_{\mathbf{f}}(\ell)$ can be constructed as an extension of the branched covering $N_{\mathbf{f}}(\ell) \to N_{\mathbf{f}}(\ell-1)$, which is known by the induction hypothesis.

Now passing to the union as $\ell \to \infty$, we conclude that the conformal conjugacy class of $\mathbf{f} : \mathcal{U}_{\mathbf{f}} \to \mathcal{U}_{\mathbf{f}}$ is uniquely determined. Passing to the Blaschke product model, this means that the associated point in $\mathcal{B}^{\mathbf{w}+1}$ is uniquely determined, up to a choice of boundary markings. But the boundary marking must vary smoothly with $\mathbf{f}$, hence it is uniquely determined by the boundary marking for $\mathbf{f}_1$. Finally, using Theorem 5.1, it follows that $\mathbf{f}$ is uniquely determined. Since a holomorphic map which is one-to-one must be biholomorphic, this completes the proof for the case that the full mapping scheme for H_α has only one vertex.

The proof for an arbitrary connected mapping scheme S is similar. Again let $\mathcal{U}_{\mathbf{f}}$ be the union of the Fatou components which contain critical or postcritical points. Let m be the period of the unique attracting orbit, and let $\lambda_{\mathbf{f}}$ be its multiplier. Choosing some m-th root $\lambda_{\mathbf{f}}^{1/m}$, the modified Kœnigs equation

$$\kappa_{\mathbf{f}}\big(\mathbf{f}(z)\big) \;=\; \lambda_{\mathbf{f}}^{1/m}\kappa_{\mathbf{f}}(z)$$

has a solution $\kappa_{\mathbf{f}} : \mathcal{U}_{\mathbf{f}} \to \mathbb{C}$ which is unique up to a multiplicative constant. As before we can normalize so that $\kappa(c_{\mathbf{f}}^1) = +1$. Constructing open sets

$$N_{\mathbf{f}}(0) \subset N_{\mathbf{f}}(1) \subset \cdots$$

with union $\mathcal{U}_{\mathbf{f}}$ as before, we can again prove inductively that the conformal conjugacy class of $\mathbf{f}$ restricted to each $N_{\mathbf{f}}(\ell)$ is determined by the w coordinates

$$\lambda_{\mathbf{f}}^{1/m}\,,\ \kappa_{\mathbf{f}}(c_{\mathbf{f}}^2)\,,\ \ldots\,,\ \kappa_{\mathbf{f}}(c_{\mathbf{f}}^{\mathbf{w}})\,,$$

and therefore that $\mathbf{f}$ is uniquely determined. The result in the case of a scheme S with several components then follows easily, applying this argument to one component of S at a time. This completes the proof of Lemma 6.5. $\square$

PROOF OF THEOREM 6.1. It now follows that the diffeomorphism $H_\alpha \to H_\beta$ is holomorphic everywhere. In fact the Cauchy-Riemann equations, which are necessary and sufficient conditions for a C^1-smooth map to be holomorphic, are satisfied throughout a dense open subset of H_α. Hence Theorem 6.1 follows by continuity. $\square$

REMARK 6.7. In fact, the entire holomorphic dynamics of the family of maps $\mathbf{f}_{\mathcal{U}_\mathbf{f}} : \mathcal{U}_\mathbf{f} \to \mathcal{U}_\mathbf{f}$ with $\mathbf{f} \in H$ depends only on the mapping scheme S. More precisely, let $\mathcal{U}_H \subset H \times |S_0| \times \mathbb{C}$ be the set of triples $(\mathbf{f},\ s,\ z)$ with $\mathbf{f} \in H$ and $(s,\ z) \in \mathcal{U}_\mathbf{f}$. Then the biholomorphic conjugacy class of the dynamical system $(\mathbf{f},\ s,\ z) \mapsto \big(\mathbf{f},\ \mathbf{f}(s,\ z)\big)$ depends only on the mapping scheme of H. The proof is similar to the proof of Theorem 6.1.

REMARK 6.8. The above discussion doesn't discuss boundary behavior. In fact, the diffeomorphism of Theorem 6.1 cannot always extend continuously over the boundary. (Compare Example 9.8.)

Here is an even deeper question. In the Douady-Hubbard theory of the Mandelbrot set M, every hyperbolic component $H \subset M$ embeds in a small copy $M_H \subset M$, where M_H is homeomorphic to M under a homeomorphism which carries $H \subset M_H$ to the cardioid component $H_0 \subset M$. More generally, we can ask the following.

> *Under what conditions does the canonical biholomorphic map from $H_0^S \subset \mathcal{P}^S$ to a given hyperbolic component $H \subset \mathcal{P}^{S_0}$ extend to an embedding of the entire connectedness locus $\mathcal{C}(\mathcal{P}^S)$ into $\mathcal{C}(\mathcal{P}^{S_0})$?*

As a simplest example, let H be a hyperbolic component of type D in the cubic connectedness locus $\mathcal{P}^3$. (Compare Figure 2D.) When is H contained in a complete Cartesian product $M \times M$ of two copies of the Mandelbrot set?

7. Real Forms.

First consider a real polynomial map $f_\mathbb{R} : \mathbb{R} \to \mathbb{R}$, of degree $d \geq 2$. We can extend $f_\mathbb{R}$ uniquely to a complex polynomial map $f : \mathbb{C} \to \mathbb{C}$. This extended map will commute with the complex conjugation operation $z \mapsto \overline{z}$. If f is hyperbolic, and if $\mathcal{U}_f$ is the union of those Fatou components which contain critical points, then complex conjugation carries this set $\mathcal{U}_f$ onto itself, carrying each component which intersects the real axis onto itself, but interchanging the remaining components in pairs. In order to find an appropriate universal model for such behavior, we consider the following construction.

Given a scheme S_0, consider antilinear[4] involutions $\gamma : |S_0| \times \mathbb{C} \to |S_0| \times \mathbb{C}$ which commute with the base map

$$\mathbf{f}_0(s,\ z)\ =\ (F(s),\ z^{d(s)})\ . \tag{7.1}$$

[4]A function γ between complex vector spaces is called ***antilinear*** (or "conjugate-linear") if $\gamma(c\,v) = \overline{c}\ \gamma(v)$ for every $c \in \mathbb{C}$.

In other words, for each $s \in |S_0|$ we assume that γ is antilinear as a function from $s \times \mathbb{C}$ to some $s' \times \mathbb{C}$, and further we assume that

$$\gamma \circ \gamma = \text{identity}\,, \qquad \text{and} \qquad \gamma \circ \mathbf{f}_0 = \mathbf{f}_0 \circ \gamma\,.$$

DEFINITION 7.1. For any γ satisfying these conditions, the subset $\mathcal{P}^{S_0}(\gamma)$ consisting of all $\mathbf{f} \in \mathcal{P}^{S_0}$ which commute with γ will be called a ***real form*** of $\mathcal{P}^{S_0}$. Two such real forms $\mathcal{P}^{S_0}(\gamma_1)$ and $\mathcal{P}^{S_0}(\gamma_2)$ will be called ***isomorphic*** if there is an automorphism η of $|S_0| \times \mathbb{C}$ which is linear on each $s \times \mathbb{C}$ such that the subset $\mathcal{P}^{S_0}(\gamma_1) \subset \mathcal{P}^{S_0}$ maps isomorphically onto $\mathcal{P}^{S_0}(\gamma_2)$ under the correspondence $\mathbf{f} \mapsto \eta^{-1} \circ \mathbf{f} \circ \eta$. As an example, if $\gamma_2 = \eta^{-1} \circ \gamma_1 \circ \eta$, then the corresponding real forms will certainly be isomorphic.

The following two lemmas will help to clarify this definition.

LEMMA 7.2. *Any antilinear involution γ of $|S_0| \times \mathbb{C}$ which commutes with $\mathbf{f}_0$ is given by the formula*

$$\gamma(s,\, z) \;=\; \big(s'\,,\; a(s)\,\overline{z}\big)\,,$$

where each $a(s)$ is a root of unity satisfying

$$a\big(F(s)\big) \;=\; a(s)^{d(s)}\,, \tag{7.2}$$

and where $s \leftrightarrow s'$ is an involution of $|S_0|$ (or the identity map) satisfying

$$d(s) = d(s')\,, \quad a(s) = a(s')\,, \quad \textit{and} \quad F(s') = F(s)'\,.$$

For each scheme S_0, there are only finitely many such involutions γ.

PROOF. The fact that the $a(s)$ are roots of unity depends on following the relation (7.2) around each cycle contained in S_0. Further details are straightforward, and will be left to the reader. $\square$

We have described $\mathcal{P}^{S_0}$ as a complex affine space. However, since it has a preferred base point $\mathbf{f}_0$, there is a closely related complex vector space consisting of all differences $\mathbf{f} - \mathbf{f}_0$.

LEMMA 7.3. *As in Lemma 7.2, let γ be an antilinear involution of $|S_0| \times \mathbb{C}$ which commutes with $\mathbf{f}_0$. Then γ acts on $\mathcal{P}^{S_0}$ by an involution*

$$\mathbf{f} \;\mapsto\; \gamma \circ \mathbf{f} \circ \gamma\,, \tag{7.3}$$

which acts antilinearly on the vector spaces of differences $\mathbf{f} - \mathbf{f}_0$. The fixed point set $\mathcal{P}^{S_0}(\gamma) \subset \mathcal{P}^{S_0}$ of the involution (7.3) is a real affine space with real dimension $w(S_0)$.

Thus $\mathcal{P}^{S_0}(\gamma)$ is a real affine space whose elements are complex maps. The real dimension of $\mathcal{P}^{S_0}(\gamma)$ is equal to the complex dimension of $\mathcal{P}^{S_0}$, or in other words to half the real dimension of $\mathcal{P}^{S_0}$.

REMARK 7.4. Two different antilinear involutions γ and γ' of $|S_0| \times \mathbb{C}$ may give rise to the same antilinear involution (7.3) of $\mathcal{P}^{S_0}$, and hence to the same real form. This occurs if and only if $\gamma \circ \gamma'$ belongs to the subgroup $\mathcal{G}_0(S_0)$ of linear automorphisms which commute with all elements of $\mathcal{P}^{S_0}$, as described in Lemma 3.6.

Proof of Lemma 7.3. If

$$\mathbf{f}(s,\, z) \;=\; \Big(F(s)\,,\; z^d + \sum_0^{d-2} c_j z^j\Big)\,,$$

where $d = d(s)$, then a brief computation using Lemma 7.2 shows that

$$\gamma\circ\mathbf{f}\circ\gamma(s,\, z) \;=\; \Big(F(s)\,,\; z^d + \sum_0^{d-2} a\big(F(s)\big)\,\overline{a}(s)^j\,\overline{c}_j\, z^j\Big)\,.$$

Evidently the correspondence $c_j \mapsto a\big(F(s)\big)\,\overline{a}(s)^j\,\overline{c}_j$ between coefficients is antilinear, as asserted.

Now note that for any antilinear involution of a complex vector space V, the fixed point set is a real vector space with real dimension equal to exactly half of the complex dimension of V. In fact the underlying real vector space of V splits as the direct sum of the $(+1)$-eigenspace, which is precisely the fixed point set, and the (-1)-eigenspace. But antilinearity guarantees that multiplication by $\sqrt{-1}$ must interchange the $+1$ and -1 eigenspaces, and the conclusion follows. □

If we generalize our spaces of polynomial maps by allowing not only monic polynomials with leading coefficient $+1$, but also polynomials with leading coefficient -1, then there is an alternative standard model for real forms which is perhaps easier to work with, since it eliminates the complex roots of unity of Lemma 7.2.

Definition 7.5. Given any choice of signs $\boldsymbol{\sigma} : |S_0| \to \{\pm 1\}$, let $\mathcal{P}^{S_0}_{\boldsymbol{\sigma}}$ be the complex affine space consisting of all maps $\mathbf{f} : |S_0|\times\mathbb{C} \to |S_0|\times\mathbb{C}$ which carry each $s\times\mathbb{C}$ to $F(s)\times\mathbb{C}$ by a centered polynomial of degree $d(s)$ with leading coefficient $\boldsymbol{\sigma}(s)$.

Definition 7.6. To any automorphism $\boldsymbol{\iota}$ of S_0 which satisfies $\boldsymbol{\iota}\circ\boldsymbol{\iota} = \text{identity}$, so that $\boldsymbol{\iota}$ is either the identity map or an involution, there is associated a ***standard antilinear involution*** $\gamma_{\boldsymbol{\iota}} : |S_0|\times\mathbb{C} \to |S_0|\times\mathbb{C}$, given by

$$\gamma_{\boldsymbol{\iota}}(s,\, z) \;=\; \big(\boldsymbol{\iota}(s),\, \overline{z}\big)\,.$$

Combining this with the previous definition, we can form the real affine space $\mathcal{P}^{S_0}_{\boldsymbol{\sigma}}(\gamma_{\boldsymbol{\iota}})$ consisting of all $\mathbf{f}\in\mathcal{P}^{S_0}_{\boldsymbol{\sigma}}$ with

$$\mathbf{f}\circ\gamma_{\boldsymbol{\iota}} \;=\; \gamma_{\boldsymbol{\iota}}\circ\mathbf{f}\,.$$

Lemma 7.7. *Let $\mathcal{P}^{S_0}(\gamma)$ be a real form of $\mathcal{P}^{S_0}$, and let $\boldsymbol{\iota} : s \leftrightarrow s'$ be the automorphism of S_0 associated with γ. Then there exists at least one choice of signs $\boldsymbol{\sigma} : |S_0| \to \{\pm 1\}$ so that the real form $\mathcal{P}^{S_0}_{\boldsymbol{\sigma}}(\gamma_{\boldsymbol{\iota}}) \subset \mathcal{P}^{S_0}_{\boldsymbol{\sigma}}$ is isomorphic to the real form $\mathcal{P}^{S_0}(\gamma)\subset\mathcal{P}^{S_0}$.*

Proof. We will construct an automorphism η of $|S_0|\times\mathbb{C}$ of the form

$$\eta(s,\, z) \;=\; \big(s,\, b(s)\, z\big)$$

and an associated choice of signs $\boldsymbol{\sigma}$ so that the conjugation $\mathbf{f}\mapsto\eta^{-1}\circ\mathbf{f}\circ\eta$ maps $\mathcal{P}^{S_0}(\gamma)$ isomorphically onto $\mathcal{P}^{S_0}_{\boldsymbol{\sigma}}(\gamma_{\boldsymbol{\iota}}) \subset \mathcal{P}^{S_0}_{\boldsymbol{\sigma}}$. Start with the expression

$$\gamma(s,\, z) \;=\; \big(s',\; a(s)\,\overline{z}\big)$$

of Lemma 7.2, and choose each $b(s)$ so that $b(s)^2 = a(s)$ with $b(s') = \overline{b(s)}$. Then a brief computation, using the fact that $|b(s)| = 1$, shows that

$$\eta^{-1} \circ \gamma \circ \eta = \gamma_{\boldsymbol{\iota}} .$$

Now define $\boldsymbol{\sigma}(s)$ by the equation

$$\boldsymbol{\sigma}(s)\, b\big(F(s)\big) = b(s)^{d(s)} . \tag{7.4}$$

Squaring this equation (7.4), and using the identity $a\big(F(s)\big) = a(s)^{d(s)}$, we see that $\boldsymbol{\sigma}(s) = \pm 1$. For the base map of equation (7.1), a straightforward computation shows that

$$\eta^{-1} \circ \mathbf{f}_0 \circ \eta(s,\, z) = \Big(F(s)\,,\ b(F(s))^{-1} b(s)^{d(s)} z^{d(s)}\Big) = \Big(F(s),\, \boldsymbol{\sigma}(s)\, z^{d(s)}\Big) .$$

A similar argument shows that the correspondence $\mathbf{f} \mapsto \eta^{-1} \circ \mathbf{f} \circ \eta$ carries $\mathcal{P}^{S_0}$ isomorphically onto $\mathcal{P}^{S_0}_{\boldsymbol{\sigma}}$; and hence maps $\mathcal{P}^{S_0}(\gamma)$ isomorphically onto $\mathcal{P}^{S_0}_{\sigma}(\gamma_{\boldsymbol{\iota}})$. $\square$

Remark 7.8. The signs $\boldsymbol{\sigma}(s)$ are far from uniquely determined, since we are free to switch the signs of the $b(s)$. Examining the defining equation (7.4), we see that replacing any given $b(s_0)$ by $-b(s_0)$ will have the following effect:

- If $F(s) = s_0$ with $s \ne s_0$, then $\boldsymbol{\sigma}(s)$ will change sign.
- If $F(s_0) \ne s_0$ and $d(s_0)$ is odd, or if $F(s_0) = s_0$ and $d(s_0)$ is even, then $\boldsymbol{\sigma}(s_0)$ will change sign.

In all other cases, the signs remain unchanged. Here is an easy example.

Lemma 7.9. *If S_0 is a union of cycles, so that F maps S_0 bijectively onto itself, and if all the degrees $d(s)$ are even, then we can choose $\boldsymbol{\sigma}(s)$ to be identically $+1$, so that every real form $\mathcal{P}^{S_0}(\gamma)$ is isomorphic to the associated $\mathcal{P}^{S_0}(\gamma_{\boldsymbol{\iota}})$.*

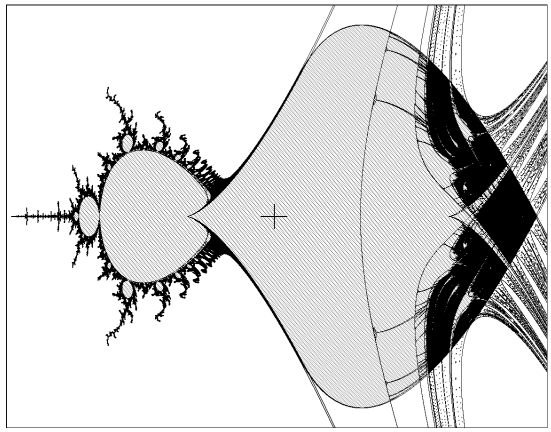
The proof is straightforward. $\square$

As another example, consider the space $\mathcal{P}^d$ of monic centered polynomial maps of degree d.

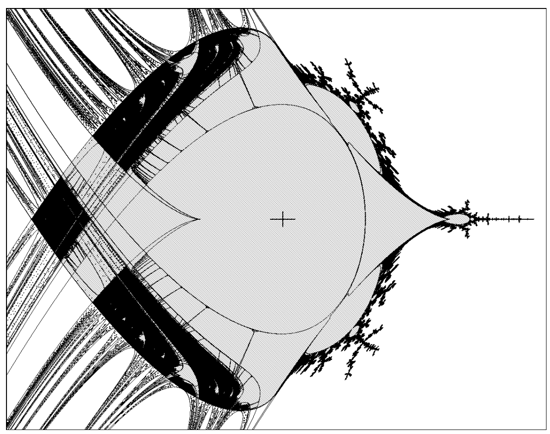
Lemma 7.10. *If the degree d is even, then every real form of $\mathcal{P}^d$ is isomorphic to the standard real form consisting of monic centered polynomials with real coefficients. But if $d \ge 3$ is odd then there are two equivalence classes, represented by real polynomials either with leading coefficient $+1$ or with leading coefficient -1.*

Proof. The associated scheme S has only one point, so the correspondence $s \leftrightarrow s'$ must be the identity map $\boldsymbol{\iota}_0$. If d is even, then it follows from Lemma 7.9 that there is only one real form $\mathcal{P}^d_+(\gamma_{\boldsymbol{\iota}_o})$ consisting of monic centered polynomials with real coefficients. However, if d is odd there is an additional real form $\mathcal{P}^d_-(\gamma_{\boldsymbol{\iota}_0})$, consisting of centered polynomials with leading coefficient -1 and with real coefficients. To see that these two are not isomorphic, note that each can be considered as a family of maps from $\mathbb{R}$ to $\mathbb{R}$. In either case, we can compactify $\mathbb{R}$ by adding points at $+\infty$ and $-\infty$, and then extend to a self-map of $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. In the $\mathcal{P}^d_+(\gamma_{\boldsymbol{\iota}_o})$ case we obtain two fixed points at infinity; but in the $\mathcal{P}^d_-(\gamma_{\boldsymbol{\iota}_o})$ case we obtain a period two orbit at infinity. $\square$

Compare Figure 5 for pictures of the connectedness locus in the two real forms of $\mathcal{P}^3$, denoted by $\mathcal{P}^3_+(\gamma_{\boldsymbol{\iota}_0})$ and $\mathcal{P}^3_-(\gamma_{\boldsymbol{\iota}_0})$. These are 2-dimensional pictures, since the total weight is $\mathbf{w} = 2$.



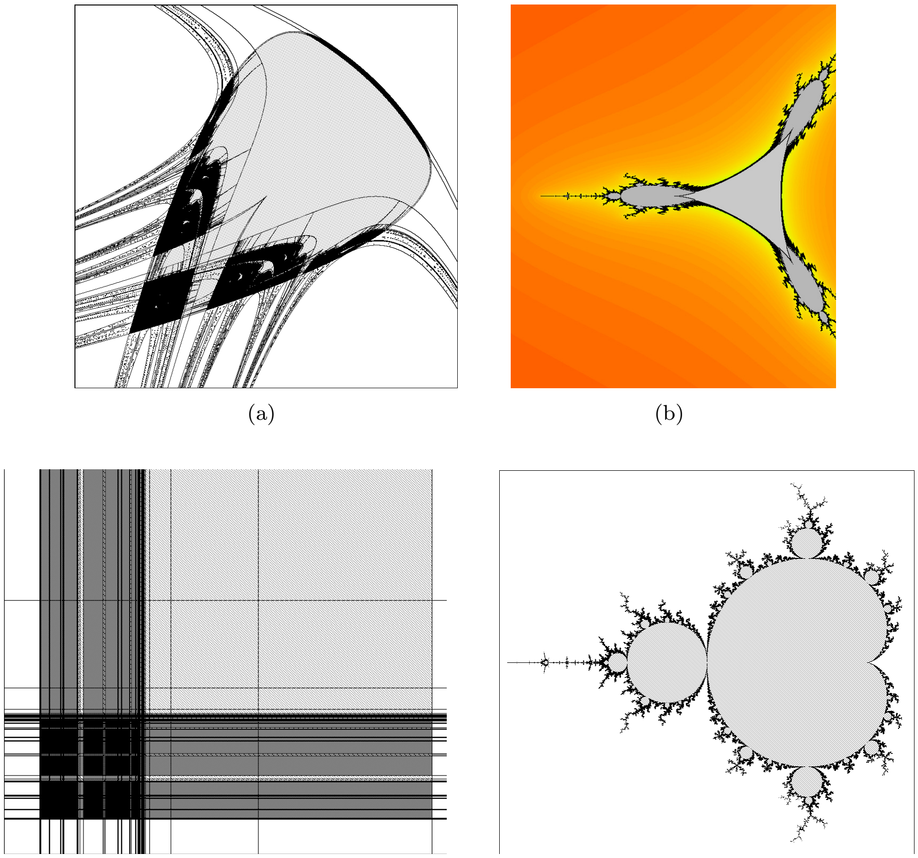
(a)



(b)

FIGURE 5. *The spaces $\mathcal{P}^3_+(\gamma_{\iota_0})$ and $\mathcal{P}^3_-(\gamma_{\iota_0})$ of real cubic maps, intersected with the complex connectedness locus $\mathcal{C}^3$. More precisely, these pictures show the real (A, b)-plane where $x \mapsto \pm x^3 - 3Ax + b$.* (*Compare* [**M1**].) *In both figures there is a compact real connectedness locus, containing Mandelbrot-like subsets in the left (respectively right) half-plane and also a Cantor set's worth of curves reaching off to infinity in the other half-plane. These curves represent maps for which just one of the two critical points has bounded orbit.*

EXAMPLE 7.11 (OTHER SCHEMES OF WEIGHT TWO). Let S_B be the ***bitransitive*** scheme, as illustrated in Figure 2B. Thus the two points s_1 and s_2 of $|S_B|$ are mapped to each other: $F : s_1 \leftrightarrow s_2$. It follows from Lemma 7.9 that there are just two real forms $\mathcal{P}^{S_B}(\gamma_{\iota})$ corresponding to the two possible choices of $\iota : s \leftrightarrow s'$. In either case, every periodic orbit must have even period.



(a) (b) (c) (d)

FIGURE 6. *Connectedness loci for four real forms of weight two.*

The Top.[5] Choosing the identity map $\boldsymbol{\iota}_0(s) = s$, we obtain the antilinear map $\gamma_{\boldsymbol{\iota}_0}$ with

$$(s_1, z) \leftrightarrow (s_1, \overline{z}), \qquad (s_2, z) \leftrightarrow (s_2, \overline{z}). \tag{7.5}$$

The corresponding real parameter space $\mathcal{P}^{S_B}(\gamma_{\boldsymbol{\iota}_0})$ can be described as the set of all maps of the form

$$\mathbf{f}(s_1, z) \ = \ (s_2, z^2 + c_1) \qquad \mathbf{f}(s_2, w) \ = \ (s_1, w^2 + c_2)$$

where c_1, c_2 are real parameters. Note that the 2-fold iterate has the form

$$\mathbf{f}^{\circ 2}(s_1, z) \ = \ \big(s_1, z^4 + 2c_1 z^2 + (c_1^2 + c_2)\big),$$

where the expression on the right varies precisely over all even real polynomials of degree 4. (Compare [**Ra**].) The connectedness locus in the (c_1, c_2)-plane is shaped like a children's top, as shown in Figure 6(a). For each $\mathbf{f}$ in the connectedness locus,

[5] "Top" in the sense of a children's spinning toy (trompo, toupie, Kreisel). (This figure might also be interpreted as an alien face, with pointy chin and shiny eyes.)

the union of all real bounded orbits consists of a non-trivial closed interval in each $s_j \times \mathbb{R}$,

$$K_{\mathbb{R}}(\mathbf{f}) \;=\; \big(s_1 \times [-a_1,\, a_1]\big) \;\cup\; \big(s_2 \times [-a_2,\, a_2]\big)\;,$$

where the right hand endpoints form a periodic orbit $(s_1,\, a_1) \;\leftrightarrow\; (s_2,\, a_2)$ with multiplier $\lambda \geq 1$. The boundary of the connectedness locus consists of three real analytic pieces, which are defined respectively by the conditions that $c_1 = -a_2$, or $c_2 = -a_1$, or that $\lambda = 1$. (In the first two cases, one critical orbit is preperiodic, with $\mathbf{f}^{\circ 2}(s_j,\, 0) = (s_j, a_j)$), while in the third case both critical orbits converge to the same parabolic orbit.) At the common endpoint of any two of these three real analytic pieces, two of these three conditions are satisfied.

As in Figure 5, the parameter picture also shows uncountably many curves reaching off to infinity, representing maps for which just one of the two critical points has bounded orbit.

The Tricorn. The other real form of $\mathcal{P}^{S_B}$ corresponds to the non-trivial involution $\boldsymbol{\iota}_1 : s_1 \leftrightarrow s_2$, with

$$\gamma_{\boldsymbol{\iota}_1} \;:\; (s_1,\, z) \;\longleftrightarrow\; (s_2,\, \overline{z})\,. \tag{7.6}$$

The form $\mathcal{P}^{S_B}(\gamma_{\boldsymbol{\iota}_1})$ then consists of all maps of the form

$$\mathbf{f}(s_1,\, z) \;=\; (s_2,\, z^2 + c)\,, \qquad \mathbf{f}(s_2,\, w) \;=\; (s_1,\, w^2 + \overline{c}) \tag{7.7}$$

where the parameter c is complex. The corresponding connectedness locus, as shown in Figure 6(b), is known as the ***tricorn*** (or ***Mandelbar set***). (Compare [**CHRS**], [**M1**], [**NS2**].) The tricorn is invariant under 120° rotation. To prove this note that the equation (7.7) remains valid if $z,\, w,\, c$ are replaced by $\eta\, z,\; \overline{\eta}\, w$ and $\overline{\eta}\, c$ respectively, where $\eta^3 = 1$. The central hyperbolic component of the tricorn, to be denoted by $\mathcal{H}_{\rm tric}$, is bounded by a deltoid curve. In fact the closure $\overline{\mathcal{H}}_{\rm tric}$ can be parametrized as

$$c(t) \;=\; t/2 \;-\; \overline{t}^{\,2}/4\;,$$

where t varies over the closed unit disk $\overline{\mathbb{D}}$. (Note that $c(\eta\, t) = \eta\, c(t)$ when $\eta^3 = 1$.) There are cusps at the three points where $t^3 = -1$. For each $t \in \overline{\mathbb{D}}$ with $t^3 \neq -1$, the Julia set consists of two simple closed curves, each mapped to the other with degree two. Thus we can parametrize one of these curves by the circle $|z| = 1$ so that the second iterate maps z to z^4, It follows that there are three period two orbits in the Julia set, corresponding to the three cube roots of unity. For $|t| < 1$ there is also an attracting period two orbit

$$(s_1,\, \overline{t}/2) \;\leftrightarrow\; (s_2, t/2)$$

in the Fatou set, with multiplier $t\,\overline{t} \in [0,\, 1)$. As $|t| \to 1$, this attracting orbit tends to one of the repelling period two orbits in the Julia set, and these become parabolic in the limit. The three edges of $\partial\mathcal{H}_{\rm tric}$ correspond to the three cube roots of unity in the discussion above. However, at the three cusp points, two of the three boundary period two orbits crash together so that each component of the Julia set becomes a copy of the "fat basilica" Julia set $J(z \mapsto z^2 - z)$.

At each of the three cusps of $\overline{\mathcal{H}}_{\rm tric}$, there is an attachment which resembles a distorted copy of the 1/2-limb of the Mandelbrot set. In fact the intersection of the left hand attachment with the real axis is identical to the real part of the 1/2-limb. However, the distortion is so extreme that the tricorn is not locally connected. (See [**NS1**], [**HS**].) Furthermore, the resemblance is not perfect. In particular, for any

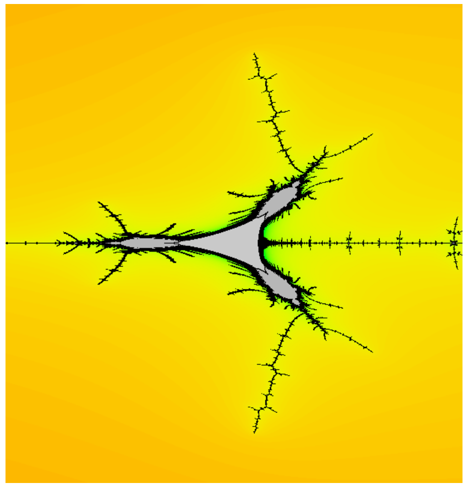


FIGURE 7. *A small copy of the tricorn contained in the tricorn, centered at $c = -1.7548\cdots$ along the real axis.*

primitive small copy of the Mandelbrot set with odd period within the Mandelbrot 1/2-limb there is a corresponding small copy[6] of the tricorn within the tricorn. (Compare Figure 7.)

It is interesting to note that both Figures 5(a) and 5(b) also contain small distorted copies of the tricorn, which would be visible only under high magnification. (Compare [**M1**], [**NS2**], and see Figure 8.)

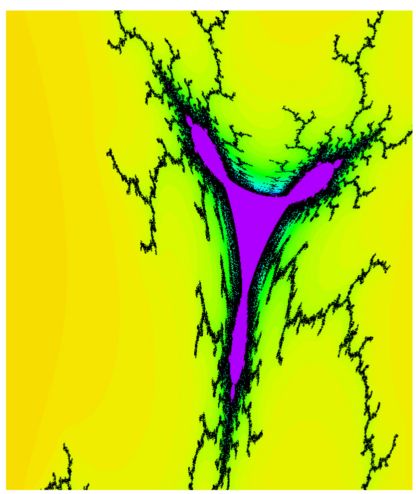


FIGURE 8. *Detail of Figure 5(a), showing a small tricorn. (Window: $[-.627, -.62] \times [.47, .477]$ in the real (A, b)-plane.)*

Decomposable Cases. Next let S_D be the ***decomposable*** scheme of weight two, as illustrated in Figure 2D. The identity correspondence $\boldsymbol{\iota}_0(s) = s$ gives rise

[6] Here the word "copy" is used loosely, and does not necessarily mean homeomorphic copy.

to a real form $\mathcal{P}^{S_D}(\gamma_{\iota_0})$ consisting of pairs of non-interacting quadratic maps

$$(s_1,\, z) \;\mapsto\; (s_1,\, z^2 + c_1)\,, \quad (s_2,\, w) \;\mapsto\; (s_2,\, w^2 + c_2)$$

with real coefficients c_1 and c_2. The corresponding connectedness locus in the real $(c_1,\, c_2)$-plane is just the Cartesian product $[-2,\, 1/4] \times [-2,\, 1/4]$ of two copies of the real part of the Mandelbrot set: See Figure 6(c), where the chaotic region is shown in black. Similarly, the nontrivial involution $\iota_1 : s_1 \leftrightarrow s_2$ gives rise to a real form $\mathcal{P}^{S_D}(\gamma_{\iota_1})$ consisting of pairs of complex holomorphic maps which are complex conjugates of each other

$$(s_1,\, z) \;\mapsto\; (s_1,\, z^2 + c)\,, \qquad (s_2,\, w) \;\mapsto\; (s_2,\, w^2 + \overline{c})\,.$$

In this case, the connectedness locus in the c-plane is just the Mandelbrot set.

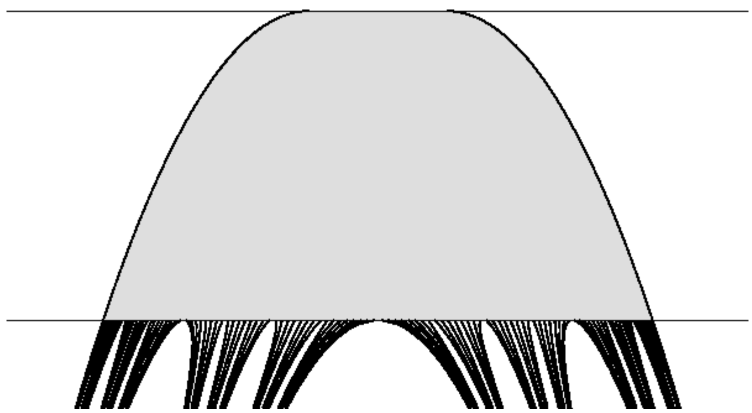


FIGURE 9. *Connectness locus for the real form $\mathcal{P}^{S_C}(\gamma_{\iota_0})$*

The Capture Component. Finally, the ***capture*** scheme S_C of Figure 2C, with

$$F : s_1 \mapsto s_2 \mapsto s_2\,,$$

has a unique real form $\mathcal{P}^{S_C}_{+,+}(\gamma_{\iota_0})$ consisting of maps

$$(s_1,\, z) \;\mapsto (s_2,\, z^2 + c_1) \qquad (s_2,\, w) \;\mapsto\; (s_2,\, w^2 + c_2)\,, \tag{7.8}$$

with real $c_1,\, c_2$. The corresponding connectedness locus in the $(c_1,\, c_2)$-plane is the shaded region in Figure 9. It can be described as the set of all real pairs $(c_1,\, c_2)$ such that both c_1 and c_2 have bounded orbit under the map $w \mapsto w^2 + c_2$. In fact c_2 has bounded orbit if and only if $-2 \le c_2 \le 1/4$ (corresponding to the region between the two parallel lines in the figure), while c_1 has bounded orbit for points in the shaded region, and also for points along the uncountable family of curves leading off to infinity below this shaded region.

REMARK 7.12. If we rely only on Remark 7.8, then we would expect to find a second real form $\mathcal{P}^{S_C}_{-,+}(\gamma_{\iota_0})$ consisting of maps

$$(s_1,\, z) \;\mapsto (s_2,\, -z^2 + c_1) \qquad (s_2,\, w) \;\mapsto\; (s_2,\, w^2 + c_2) \tag{7.9}$$

with real $c_1,\, c_2$. However, the two forms (7.8) and (7.9) are actually isomorphic under the complex change of coordinates

$$\eta(s_1,\, z) = (s_1,\, iz)\,, \quad \eta(s_2,\, w) = (s_2,\, w)\,.$$

On the other hand, if we consider (7.8) and (7.9) simply as defining maps from $|S| \times \mathbb{R}$ to itself, and allow only real changes of coordinate, then these two forms really are non-isomorphic. This discrepancy between real coordinate changes and complex coordinate changes is closely related to Remark 7.4.

To conclude, note the following analogue of the results in §5 and §6.

THEOREM 7.13. *Every hyperbolic component in a real connectedness locus of weight* $\mathbf{w}$ *is a topological* $\mathbf{w}$*-cell with a unique "center" point, and is real analytically homeomorphic to a uniquely defined principal hyperbolic component* $H_0^S(\gamma)$*, or to a suitably defined space* $B^S(\gamma)$ *of Blaschke products, under a homeomorphism which is uniquely determined up to the action of the appropriate finite symmetry group.*

The proof involves going through the arguments in previous sections, keeping track of the extra involution, and is not difficult. $\square$

8. Polynomials with Marked Critical Points.

By a ***critically marked*** polynomial map of degree $\mathbf{w}+1$ we will mean a polynomial map f together with an ordered list $(c_1, \ldots, c_{\mathbf{w}})$ of its critical points. Even if f is a real polynomial, this list must include all complex critical points, with a critical point of multiplicity m listed m times, so that the derivative is given by

$$f'(z) = (\mathbf{w}+1)(z-c_1)\cdots(z-c_{\mathbf{w}}),$$

whenever f is a monic. As an example, Branner and Hubbard [**BH**] studied critically marked cubic polynomials, using the monic centered normal form

$$f(z) = z^3 - 3a^2 z + b,$$

with ordered list of critical points $(a, -a)$.

Similarly, we can define the concept of a critically marked Blaschke product. All of the principal results of the previous sections extend to the marked case. In particular, for any mapping scheme S we can define a space $\mathcal{P}^S_{\mathrm{cm}}$ of marked polynomial maps and a space $\mathcal{B}^S_{\mathrm{cm}}$ made up out of marked Blaschke products. Then any hyperbolic component of type S' in the marked connectedness locus $\mathcal{C}^S_{\mathrm{cm}} \subset \mathcal{P}^S_{\mathrm{cm}}$ is canonically homeomorphic to $\mathcal{B}^{S'}_{\mathrm{cm}}$. The one step in this program which causes additional difficulty is the analogue of Lemma 4.9, showing that the appropriate spaces of critically marked Blaschke products are topological cells. For this we need the following result. (Compare [**Bo**], [**Z**].)

THEOREM 8.1 (Bousch, Zakeri). *A Blaschke product* $\beta : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *of degree* $d = \mathbf{w}+1$ *which fixes the points* 0 *and* 1 *is uniquely determined by its critical points* $c_1, \ldots, c_{\mathbf{w}}$*, which can be arbitrary points of the open unit disk. Hence the space of all such maps is diffeomorphic to the* $\mathbf{w}$*-fold symmetric product of* $\mathbb{D}$ *with itself. In particular, this space is a topological cell of dimension* $2\mathbf{w}$*.*

As a substitute for zeros centered maps in the critically marked case, it seems natural to work with the space of 1-anchored critically marked Blaschke products which are ***critically centered*** in the sense that $c_1 + c_2 + \cdots + c_{\mathbf{w}} = 0$. However, in order to determine such a Blaschke product β uniquely, we need one more piece of information, namely the value $\beta(0)$. (Compare the statement that a monic polynomial is uniquely determined by its critical points together with its constant term.) Using the Bousch-Zakeri theorem, is not difficult to check that the space of all such critically marked and centered Blaschke products of degree $\mathbf{w}+1$ is a topological $2\mathbf{w}$-cell. (For the special case $\mathbf{w}=0$, this definition doesn't make sense, so we simply define the corresponding space of Blaschke products to consist of the identity map only.)

It then follows easily that the corresponding model space $\mathcal{B}^S_{\rm cm}$, made out of critically marked Blaschke products which are either fixed point centered or critically center is also a topological cell.

The concept of a ***real form*** for the space $\mathcal{P}^S_{\rm cm}$ can be defined in analogy with the discussion in §7. In general, the space $\mathcal{P}^S_{\rm cm}$ has more real forms than $\mathcal{P}^S$. (This is closely related to the fact that $\mathcal{P}^S_{\rm cm}$ has more symmetries than $\mathcal{P}^S$.) As an example, the space $\mathcal{P}^3_{\rm cm}$ of critically marked cubic maps has four distinct real forms, which can be put in the form

$$f(z) \;=\; \pm(z^3-3a^2z+b)\,.$$

Here the critical points $\{a,\,-a\}$ are either real or pure imaginary, and the sign is either $+$ or $-$, while b is always real. More generally, for $2d-1\geq 3$ the space $\mathcal{P}^{2d-1}_{\rm cm}$ has $2d$ distinct real forms which can be labeled by the initial sign together with the number of real critical points. Similarly $\mathcal{P}^{2d}_{\rm cm}$ has d real forms.

9. Rational Maps.

Many of the constructions from this note can be applied also to hyperbolic rational maps $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$. However, there is a difficulty with the statements, since there is no convenient normal form which works for all rational maps, and a difficulty with the proofs since the boundary of a critical hyperbolic Fatou component need not be a Jordan curve. We can deal with the first problem by introducing a suitable moduli space. (Compare [**M3**].)

Definition 9.1. By a ***fixedpoint-marked rational map*** $(f;\,z_0,\,z_1,\,\ldots,\,z_d)$ will be meant a rational map $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ of degree $d\geq 2$, together with an ordered list of its $d+1$ (not necessarily distinct) fixed points z_j.

Lemma 9.2. *The space* $\mathrm{Rat}^d_{\rm fm}$ *of all such fixedpoint-marked maps of degree* d *is a smooth manifold of complex dimension* $2d+1$.

Proof. First consider the open subset consisting of all points of $\mathrm{Rat}^d_{\rm fm}$ with $f(\infty)\neq\infty$. Each such $f(z)$ can be written uniquely as a quotient $p(z)/q(z)$ of two polynomials with $q(z)$ monic of degree d. The fixed point equation then takes the form

$$z\,q(z)\,-\,p(z)\;=\;(z-z_0)(z-z_1)\cdots(z-z_d)\;, \tag{9.1}$$

where the z_j are the (not necessarily distinct) fixed points of f. Given $q(z)$ and the $z_j\in\mathbb{C}$, we can solve uniquely for $p(z)$. Here $p(z)$ will be relatively prime to $q(z)$ if and only if $q(z_j)\neq 0$ for all j. Thus we have a well behaved coordinate neighborhood. Similarly, for each integer $0\leq n\leq d$ the set of f satisfying $f(n)\neq n$ is a coordinate neighborhood. The entire space is covered by these $d+2$ coordinate neighborhoods, since a map of degree d can have at most $d+1$ fixed points. □

If we conjugate such a fixedpoint-marked rational map by a Möbius automorphism $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, then we obtain a new fixedpoint-marked map

$$(g \circ f \circ g^{-1}\,;\; g(z_0),\, g(z_1),\, \ldots,\, g(z_d))\ .$$

The quotient space of $\mathrm{Rat}^d_{\mathrm{fm}}$ under this action of the Möbius group will be called the ***moduli space*** $\mathcal{M}^d_{\mathrm{fm}}$ for fixedpoint-marked maps. This moduli space is a noncompact complex algebraic variety of dimension $2d-2$. The action of the Möbius group is clearly free on the open subset consisting of points of $\mathrm{Rat}^d_{\mathrm{fm}}$ with at least three distinct fixed points. Thus $\mathcal{M}^d_{\mathrm{fm}}$ has possible singularities only within the subvariety consisting of conjugacy classes with at most two distinct fixed points.

By a ***hyperbolic component*** in $\mathcal{M}^d_{\mathrm{fm}}$ will be meant a connected component in the open subset consisting of all conjugacy classes of hyperbolic fixedpoint-marked maps. Each such hyperbolic component is a smooth complex manifold, since every hyperbolic map has $d+1$ distinct fixed points. By definition, such a hyperbolic component belongs to the **connectedness locus** if its representative maps have connected Julia set.

THEOREM 9.3. *Every hyperbolic component $\mathcal{H}$ in the connectedness locus of $\mathcal{M}^d_{\mathrm{fm}}$ is canonically homeomorphic to the model space $\mathcal{B}^S$, where $S = S_f$ is the mapping scheme for a representative map f. In particular, every such $\mathcal{H}$ is simply connected,*[7] *with a unique critically finite point. Similarly, $\mathcal{H}$ is biholomorphic to the standard model of Definition 6.2.*

The proof will be based on the following preliminary result. Consider rational maps with just three marked fixed points. If these three points are distinct, then there is a unique Möbius conjugate with these fixed points respectively at zero, one, and infinity. The resulting map can be written uniquely as a quotient $f(z) = p(z)/q(z)$ of two relatively prime polynomials, with $p(0) = 0$ and $p(1) = q(1)$, where $p(z)$ is monic of degree d, and where $q(z)$ has degree at most $d-1$. It follows easily that polynomials of this form can be parametrized by an open subset of the coordinate space $\mathbb{C}^{2d-2}$.

THEOREM 9.4. *Let H be a connected component in the space of all hyperbolic rational maps of degree d which have this normal form, with fixed points at zero, one and infinity, and which have connected Julia set. Then H is homeomorphic to the model space $\mathcal{B}^S$, where S is the mapping scheme for a representative map in H.*

For the proof of this preliminary theorem, we will need a concept of "*boundary marking*" for each map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ belonging to H. However, since the topological boundary of a Fatou component U for f may not be a Jordan curve, we must work with a modified concept of boundary.

LEMMA 9.5. *Any Riemann surface U which is conformally isomorphic to the open disk $\mathbb{D}$ can be canonically embedded into an* ***ideal compactification*** $\widehat{U}$ *which is diffeomorphic to the closed disk $\overline{\mathbb{D}}$, and which is natural in the sense that any proper holomorphic map $U \to U'$ between two such Riemann surfaces extends to a smooth map $\widehat{U} \to \widehat{U}'$. In the special case where U is an open subset of the Riemann sphere $\widehat{\mathbb{C}}$, then the boundary $\partial\widehat{U}$ (the* ***ideal boundary*** *of U) can be identified with*

[7] Here it is essential that we work with fixedpoint-marked maps. Compare [**Mc2**], which exploits the fact that hyperbolic components in the space Rat^d of all (unmarked) degree d rational maps often have an interesting fundamental group.

the set of prime ends of U. In particular, whenever the topological closure $\overline{U} \subset \widehat{\mathbb{C}}$ is locally connected, the identity map of U extends to a continuous map from $\widehat{U}$ onto $\overline{U}$.

PROOF. (Compare [**Mc1**].) Choosing any conformal isomorphism $\phi : U \to \mathbb{D}$, we can pull the Euclidean metric of the unit disk back to U, and then form the metric completion $\widehat{U}$. It is not difficult to check that this completion is independent of the choice of ϕ. In fact, using Lemma 4.1, we see easily that the differentiable structure of $\overline{\mathbb{D}}$, pulled back to $\widehat{U}$, is independent of the choice of ϕ. Similarly, if U' is another Riemann surface conformally equivalent to $\mathbb{D}$, it follows from Lemma 4.1 that any proper holomorphic map $U \to U'$ extends to a smooth map between ideal compactifications. The final properties, in the case where U is an open subset of the Riemann sphere follow easily from Carathéodory theory. (See for example [**M4**, §17.13].) □

PROOF OF THEOREM 9.4. The argument now proceeds just as in the polynomial case. Choose a basepoint $f_1 \in H$ and form the space $\widetilde{H}$ consisting of triples $(f,\ q,\ \boldsymbol{\iota})$, where $f \in H$, where q is a boundary marking sending each critical or postcritical Fatou component U to a point $q(U) \in \partial\widehat{U}$ with $q\big(f(U)\big) = f\big(q(U)\big)$, and where $\boldsymbol{\iota}$ is an isomorphism from $S = S_{f_1}$ to S_f.

We must first show that every connected component of $\widetilde{H}$ is a (possibly trivial) covering space of H. In particular, we must show that the boundary marking q deforms continuously as we deform f. Clearly each repelling periodic point deforms continuously as we deform f. Mañé, Sad and Sullivan [**MSS**] showed that this extends to a continuous deformation of the entire Julia set. Then Slodowski [**Sl**] showed that this deformation can be extended to an isotopy of the entire Riemann sphere. In particular, the closure of each Fatou component deforms continuously, and it follows that each prime end deforms continuously. Thus the boundary marking $q(U)$ also deforms continuously. It then follows easily that every point of H has a neighborhood which is evenly covered under the projection $\widetilde{H} \to H$.

Just as in §4, each component of $\widetilde{H}$ is also a covering of the simply connected model space $\mathcal{B}^S$, and hence projects homeomorphically onto $\mathcal{B}^S$. It follows that each component of $\widetilde{H}$ contains a unique critically finite point, and hence must map homeomorphically onto H, as required. □

PROOF OF THEOREM 9.3. Given a point in the component $\mathcal{H} \subset \mathcal{M}^d_{\mathrm{fm}}$, note that the $d+1$ fixed points for a representative map are necessarily distinct. Using only the first three marked points, we can obtain a unique representative f in the corresponding component $H \subset \mathrm{Rat}^d$ of Theorem 9.4. This defines a projection $\mathcal{H} \to H$ which is clearly a covering map, since the distinct fixed points vary smoothly as we deform the map f. Since H is simply connected, it follows that this covering map is a homeomorphism. Finally, the proof of Theorem 6.1 extends easily to this more general context. □

REMARK 9.6. It is also interesting to study real forms of rational maps. (Compare [**M3**].) There are just two antiholomorphic involutions of the Riemann sphere up to Möbius conjugation, namely the complex conjugation operation with $\mathbb{R}\cup\infty$ as fixed point set, and the ***antipodal map*** $\gamma(z) = -1/\overline{z}$ which has no fixed points. A rational map commutes with complex conjugation if and only if it is a quotient of polynomials with real coefficients. The family of rational maps commuting with the antipodal map is more interesting. (See [**BBM**].) It includes many odd degree rational functions, such as $z \mapsto z^{2n+1}$, but no even degree functions. In fact any continuous map of the sphere which commutes with the antipodal map must have odd degree by a classical theorem of Borsuk.

Quadratic Rational Maps. The quadratic moduli space $\mathcal{M}^2_{\rm fm}$ can be identified with the affine variety consisting of all $(\alpha,\,\beta,\,\gamma) \in \mathbb{C}^3$ satisfying the equation

$$\alpha\beta\gamma \,-\, \alpha \,-\, \beta \,-\, \gamma \,+\, 2 \;=\; 0\;. \tag{9.2}$$

Here α, β and γ are the multipliers at the three marked fixed points. (Compare [**M3**]. For other work on quadratic rational maps, see for example [**R1**], [**R2**].)

If, for example, $\alpha\beta \neq 1$, then we can solve equation (9.2) for γ as a smooth function of α and β. On the other hand, if $\alpha\beta = 1$ then equation (9.2) reduces to the equality $\alpha + \beta = 2$, which implies easily that $\alpha = \beta = 1$; this corresponds to the case where the two corresponding fixed points crash together. *It follows that the affine variety defined by equation* (9.2) *is smooth except at the point* $\alpha = \beta = \gamma = 1$ *where all three fixed points crash together.* For a description of the singularity at this triple fixed point class, see the discussion following Theorem 9.11 below.

In the case $\alpha\beta \neq 1$, putting the first two fixed points at zero and infinity, a linear change of coordinates will put the corresponding rational map into the normal form

$$f(z) \;=\; z\frac{z+\alpha}{\beta\,z+1}\;, \tag{9.3}$$

with the third marked fixed point at $(1-\alpha)/(1-\beta)$. On the other hand, in the case $\alpha = \beta = \gamma = 1$, if we put the triple fixed point at infinity, then an affine change of coordinates will yield the normal form $f(z) = z + 1/z$.

Remark 9.7. A hyperbolic component in the connectness locus need not have compact closure within $\mathcal{M}^d_{\rm fm}$. In the quadratic case, Adam Epstein [**Ep**] has shown that a hyperbolic component H consisting of maps with two disjoint attracting cycles has compact closure if and only if neither attracting cycle is a fixed point. This closure $\overline{H}$ can be very difficult to visualize. (See the following non-compact Examples.)

Example 9.8. (**The Simplest Case**, although it is not very simple). Let $H_0 \subset \mathcal{M}^2_{\rm fm}$ be the hyperbolic component consisting of all $(\alpha,\,\beta,\,\gamma)$ satisfying (9.2) with $|\alpha| < 1$ and $|\beta| < 1$. The boundary of this component seems very difficult to visualize, although it is easily described as a semi-algebraic set. Certainly we must have $|\alpha| \leq 1$ and $|\beta| \leq 1$ for every $(\alpha,\ \beta,\ \gamma)$ in the closure $\overline{H}$, and using the holomorphic index formula it is not hard to show that $\Re(\gamma) \geq 1$. In fact, $\overline{H}$ is precisely equal to the set of all $(\alpha,\ \beta,\ \gamma)$ in $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \{\gamma\ ;\ \Re(\gamma) \geq 1\}$ which satisfy the equation (9.2). Since $|\gamma|$ is unbounded, it follows that $\overline{H}_0$ is non-compact, as noted in [**Ep**].

This closure $\overline{H}_0$ contains the singular point $\alpha = \beta = \gamma = 1$, which leads to rather bad behavior. For example, although $\overline{H}_0$ is simply connected, it acquires a free cyclic fundamental group if we remove the singular point. This awkward behavior disappears if we eliminate the singularity by passing to the 2-fold branched covering space which is branched only over the singular point. (Compare the discussion of the "totally marked" moduli space following Remark 9.10 below.) In fact, the corresponding hyperbolic component in the covering space has boundary homeomorphic to an open 3-cell.

Epstein has pointed out that there is a completely analogous example for cubic polynomial maps. If we look at the space of fixed point multipliers $(\alpha, \beta,\ \gamma)$ in this case, there is again a single relation, which now takes the form

$$3 - 2(\alpha + \beta + \gamma) + (\alpha\beta + \alpha\gamma + \beta\gamma) \ = \ 0\ . \tag{9.4}$$

If $|\alpha| < 1$ and $\beta| < 1$, then it follows from the holomorphic index formula that $|\gamma - 3/2| < 1/2$. The closure $\overline{H}$ of the hyperbolic component described in this way, can be obtained by intersecting the locus (9.4) with the product $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \overline{\mathbb{D}}_{1/2}(3/2)$ of the corresponding closed disks. Again this closure has very peculiar behavior, associated with the singularity at the triple fixed point. However, if we eliminate the singularity by passing to the 2-fold covering branched at this point, corresponding to the family of monic polynomials $z \mapsto z^3 + az^2 + \gamma z$, then $\overline{H}$ becomes a closed topological 4-cell. (For a similar example in the context of Kleinian groups, see [**Mc3**, Appendix A].)

FIGURE 10. *The a-plane for the family* $z \mapsto (1-a)/(z(z-a))$.

EXAMPLE 9.9. Such examples with semi-algebraic boundary are presumably very rare. Here is an example of a hyperbolic component with fractal boundary. Figure 9.9 shows the a-parameter plane for the family of maps

$$f_a(z) \ = \ \frac{a-1}{z(a-z)} \tag{9.5}$$

with critical points at ∞ and $a/2$, normalized so that $f_a(1) = 1$. Note the super-attractive period two orbit $\infty \leftrightarrow 0$. (Compare [**Du**] and [**T**], who use a different parametrization.) Here the value $a = 1$ (the cusp point at right center) must be excluded. In fact the multiplier $f'_a(1) = (2-a)/(a-1)$ tends to infinity as $a \to 1$. The central white region in this figure is part of the hyperbolic component centered at $f_0(z) = 1/z^2$, and consisting of maps with a bitransitive attracting orbit of period two. This component H in the a-parameter plane has fractal boundary, and has non-compact closure (for example, in the neighborhood of $a = 1$). The small white regions in the figure correspond to capture components where both critical orbits converge to the same attracting orbit, but only one critical point lies in a periodic Fatou component. The black regions correspond to everything else.

The fixed points are not marked in this family (9.5), but the fractal nature of ∂H would remain if we replace H by the corresponding hyperbolic component $H_{\rm fm}$ in the full moduli space $\mathcal{M}^2_{\rm fm}$ with marked fixed points. For all maps in this hyperbolic component, note that the Julia set is a simple closed curve separating the two Fatou components.

It is interesting to note that the closure $\overline{H}_{\rm fm} \subset \mathcal{M}^2_{\rm fm}$ is not only non-compact, but has at least three distinct ends (conjecturally exactly three). In fact, if we pass to infinity within moduli space, then at least one of the multipliers α, β, γ must have norm tending to infinity. Since none of the fixed points are attracting, it follows from equation (9.2) that the other two multipliers must remain bounded, tending towards conjugate points on the unit circle. (For example as $a \to 1$ along the real axis in the family (9.5), the other two multipliers tend to -1.) Thus there are at least three essentially different ways of tending to infinity within $\overline{H}_{\rm fm}$.

Remark 9.10. One would like to be able to form a submanifold of the moduli space $\mathcal{M}^d_{\rm fm}$ by requiring one or more critical points to be periodic of specified period. Since our fixed points are already numbered, we can easily specify a subvariety of $\mathcal{M}^d_{\rm fm}$ by requiring one or more of the *fixed* points to be critical of specified multiplicity. However, if we want a family of maps such that some critical point is periodic of period $p > 1$, then we must either restrict attention to one hyperbolic component as in Remark 5.12, or forget the fixed point marking as in Example 9.9, or else pass to a branched covering space of $\mathcal{M}^d_{\rm fm}$ by also marking this critical point.

Define a rational map of degree d to be ***totally marked*** if we have specified an ordered list, not only of its $d+1$ fixed points, but also of its $2d-2$ critical points. I will describe only the quadratic case, which is easier to deal with.

Theorem 9.11. *The moduli space $\mathcal{M}^2_{\rm tm}$ for totally marked quadratic maps is biholomorphic to the smooth simply connected affine variety V consisting of all $(x_1, x_2, x_3) \in \mathbb{C}^3$ which satisfy the equation*

$$x_1 + x_2 + x_3 \ + \ x_1 x_2 x_3 \ = \ 0 \ . \tag{9.6}$$

In terms of these coordinates, the multiplier of a representative rational map f at the k-th fixed point of f is given by

$$\lambda_k \ = \ 1 + x_h x_j \ , \tag{9.7}$$

where $\{h, j, k\}$ can be any permutation of $\{1, 2, 3\}$. If we switch the numbering of the two critical points, then all of the x_j change sign; while if we renumber the three fixed points, then the x_j are permuted and multiplied by either $+1$ or -1 according as the permutation is even or odd.

The proof will be given below.

It follows that the projection from $\mathcal{M}^2_{\rm tm}$ to $\mathcal{M}^2_{\rm fm}$ is a smooth 2-fold branched covering, branched only over at the point $x_1 = x_2 = x_3 = 0$, which maps to

$\lambda_1 = \lambda_2 = \lambda_3 = 1$. It follows easily that any hyperbolic component $H \subset \mathcal{M}^2_{\rm tm}$ projects diffeomorphically onto a corresponding hyperbolic component $H' \subset \mathcal{M}^2_{\rm fm}$.

Given any integer $p \ge 1$, we can define a subvariety $\mathrm{Per}_p(0) \subset \mathcal{M}^2_{\rm tm}$ by requiring that the first marked critical point should be periodic of period p, and a "dual" subvariety $\mathrm{Per}^*_p(0)$ by requiring that the second marked critical point has period p. As in Remark 5.12, it follows that for each hyperbolic component H which intersects one of these curves, the intersection is a topological 2-cell with a unique critically finite point.

LEMMA 9.12. *Each periodic curve* $\mathrm{Per}_p(0) \subset \mathcal{M}^2_{\rm tm}$ *is a smooth complex 1-manifold. Furthermore, for each* $p,\, q \ge 1$*, the curves* $\mathrm{Per}_p(0)$ *and* $\mathrm{Per}^*_q(0)$ *intersect transversally in a finite number of points.*

The proof is completely analogous to the corresponding proof for cubic polynomial maps, as given in [**M5**, §5]. Details will be omitted. □

However, this discussion leaves three unanswered questions:

- How can we count the number of points in this transverse intersection? (Compare [**M5**].)
- How can we compute the Euler characteristic of the curve $\mathrm{Per}_p(0)$? (Compare [**BKM**].)
- Is this curve always connected?

PROOF OF THEOREM 9.11. A totally marked quadratic map

$$(f,\, z_1,\, z_2,\, z_3,\, c_1,\, c_2)$$

is uniquely determined by the 5-tuple $(z_1,\, z_2,\, z_3,\, c_1,\, c_2)$ of fixed points and critical points. (Compare [**M3**, §6].) In fact, if we put c_1 at the origin and c_2 at infinity, then the map f takes the form

$$f(z) \;=\; \frac{az^2+b}{cz^2+d}\;. \tag{9.8}$$

Thus there is a fixed point at infinity if and only if $c = 0$, and the finite fixed points satisfy the equation $cz^3 - az^2 + dz - b = 0$. Clearly the collection of roots $z_1,\, z_2,\, z_3$ uniquely determines the point $(a : b : c : d)$ in projective 3-space, and hence uniquely determines the map f.

Now consider the three cross-ratios

$$r_h \;=\; \frac{(c_1 - z_j)(c_2 - z_k)}{(c_1 - z_k)(c_2 - z_j)}\;,$$

where $(h,\, j,\, k)$ is to be any cyclic permutation of $(1,\, 2,\, 3)$. These are clearly invariant under conformal conjugacy. In fact, they form a complete conjugacy invariant. Still using the normal form (9.8), we see that $r_h = z_j/z_k$. If another triple has the same ratios r_j then we can get one triple from the other by multiplying by some constant $\lambda \ne 0$. This corresponds to a conjugation of the form

$$f(z) \;\mapsto\; \lambda f(z/\lambda)\;.$$

(Here one has to take care with the special case that some z_j is zero and/or some z_k is infinity, but the conclusion still follows, using the fact that a double fixed point can never be a critical point. Details will be left to the reader.)

Recall that a point in the moduli space $\mathcal{M}^2_{\rm fm}$ is determined by the multipliers at the three fixed points, which we now denote by λ_1, λ_2, λ_3. Define a map from the variety V onto $\mathcal{M}^2_{\rm fm}$ by setting

$$\lambda_h \;=\; 1 + x_j\, x_k\ .$$

Then the required identity (9.2),

$$\lambda_1\,\lambda_2\,\lambda_3\;-\;\lambda_1\;-\;\lambda_2\;-\;\lambda_3\;+\;2\;=\;0\ ,$$

is easily verified. Note the identity

$$x_h^2 \;=\; 1 - \lambda_j\,\lambda_k\ . \tag{9.9}$$

In fact $1-\lambda_j\lambda_k \;=\; 1-(1+x_hx_k)(1+x_hx_j)$ which simplifies easily to $-x_h(x_k+x_j+x_hx_jx_k) \;=\; x_h^2$.

If $z_j \neq z_k$, then we can put the fixed point z_j at zero and z_k at infinity, and write the map as

$$f(z) \;=\; z\frac{z+\lambda_j}{\lambda_k\, z+1}\ ,$$

as in equation (9.3). A brief computation then shows that the two critical points $f'(z)=0$ satisfy the equation $\lambda_k z^2+2z+\lambda_j \;=\; 0$, with solution

$$c_i \;=\; \frac{-1\pm\sqrt{1-\lambda_j\lambda_k}}{\lambda_k} \;=\; \frac{-1\pm x_h}{\lambda_k}\ .$$

To fix our ideas, suppose that

$$c_1 \;=\; (-1+x_h)/\lambda_k\,, \quad c_2 \;=\; (-1-x_h)/\lambda_k\ .$$

Then the cross-ratio r_h is given by

$$r_h \;=\; \frac{c_1}{c_2} \;=\; \frac{-1+x_h}{-1-x_h} \;=\; \frac{1-x_h}{1+x_h}\ .$$

Thus the cross ratios r_h and hence the conjugacy class in $\mathcal{M}^2_{\rm tm}$ are uniquely determined by the coordinates x_1, x_2, x_3, yielding a holomorphic map from V to $\mathcal{M}^2_{\rm tm}$.

Conversely, given the r_h, we can solve for $x_h=(1-r_h)/(1+r_h)$. (Using this correspondence $r_h\leftrightarrow x_h$, note that the defining identity (9.6) for the variety V is completely equivalent to the relation

$$1 \;=\; r_1r_2r_3 \;=\; \frac{(1-x_1)(1-x_2)(1-x_3)}{(1+x_1)(1+x_2)(1+x_3)}\ ,$$

again taking special care with the cases where some r_h is zero or infinity.) This completes the proof that $\mathcal{M}^2_{\rm tm}$ is biholomorphic to the affine variety V.

As an example, if we take $c_1=0$ and $c_2=\infty$, with h, j, k in positive cyclic order, then

$$x_h \;=\; \frac{z_k-z_j}{z_k+z_j}\ ,$$

where the denominator can never be zero.

It is easy to prove that V is smooth and connected. In fact it is covered by three coordinate neighborhoods

$$V_h \;=\; \{\ (x_1,\,x_2,\,x_3)\;\in\;V\ ;\ x_j\,x_k\;\neq\;-1\ \}\ .$$

Here $V_1 \cup V_2 \cup V_3 = V$ since the equations $x_1 x_2 = x_1 x_3 = x_2 x_3 = -1$ have no simultaneous solution within V. For $(x_1, x_2, x_3) \in V_h$, we can solve uniquely for

$$x_h = \frac{-x_j - x_k}{1 + x_j\, x_k}$$

as a holomorphic function of the other two variables.

We first show that each coordinate neighborhood V_h has fundamental group $\pi_1(V_h) \cong \mathbb{Z}$. To simplify notation, note that V_h is biholomorphic to the complement of the quadratic curve

$$W = \{ (x, y) \in \mathbb{C}^2 \; ; \; x\, y = -1 \} .$$

Let $\mathcal{S} \subset \mathbb{C}^2$ be the real hypersurface consisting of all products $(\xi, \eta) = (tx, ty)$ with $(x, y) \in W$ and $t > 1$. Then the complement $\mathbb{C}^2 \smallsetminus (W \cup \mathcal{S})$ is star-shaped. That is, the line segment joining any point to the origin is completely contained in $\mathbb{C}^2 \smallsetminus (W \cup \mathcal{S})$. Any loop in $\mathbb{C}^2 \smallsetminus W$ can be perturbed until it intersects the hypersurface $\mathcal{S}$ transversally in finitely many points. The homotopy class of such a loop is determined by the number of transverse intersection points, counted with a sign of -1 or $+1$ according as the imaginary part $\Im(\xi\, \eta)$ is increasing or decreasing as (ξ, η) passes through $\mathcal{S}$. In fact we can use the star shaped property to drag the loop L down to the origin except in a small neighborhood of each intersection point (ξ_j, η_j) The part of this loop within this small neighborhood can then be deformed to a triangular loop $T(\xi_j, \eta_j)$ consisting of a line segment from the origin to $(1 \pm i\epsilon)(\xi_j, \eta_j)$, followed by a line segment to $(1 \mp \epsilon)(\xi_j, \eta_j)$, and then followed by a line segment back to the origin. Since $\mathcal{S}$ is connected, the homotopy class of this triangular loop does not depend on the particular choice of (ξ_j, η_j). Finally, the composition of two consecutive loops of opposite orientation is homotopic to the zero loop. Since a standard topological argument shows that the number of intersections, counted with sign, is a homotopy invariant, this proves that $\pi_1(\mathbb{C} \smallsetminus W) \cong \pi_1(V_h) \cong \mathbb{Z}$.

As an example, consider the loop L in V which is given by

$$\theta \mapsto (x_1, x_2, x_3) \quad \text{with} \quad x_1 = 1 + \epsilon\, e^{i\theta} , \; x_2 = -x_1 , \; x_3 = 0 .$$

Then L is homotopic to a constant in V_1 or in V_2, since we can simply let ϵ tend to zero; and yet it represents a generator of $\pi_1(V_3)$. It follows that the variety V is simply-connected. In fact, each inclusion $V_h \subset V$ induces a homomorphism from $\pi_1(V_h)$ onto $\pi_1(V)$, since it is easy to homotop any loop in V away from the locus $x_j x_k = -1$.

Further details of the proof of Theorem 9.11 are straightforward. □

Remark 9.13. The field $F \subset \mathbb{C}$ generated by the coordinates $x_j \in \mathbb{C}$ is an interesting invariant of the conjugacy class of f. It can be characterized as the smallest field such that some Möbius conjugate of f has all critical points and all fixed points within F.

Appendix A. Realizing Reduced Schemes (by Alfredo Poirier[6])

The purpose of this appendix is to prove that every reduced mapping scheme can be realized by a postcritically finite polynomial. In order to do so, we will construct an appropriate Hubbard tree that mimics the dynamics of the scheme.

For the benefit of the reader we recall briefly the main concepts involved in the construction of Hubbard trees following closely [**P**].

Given a degree $d \geq 2$ postcritically finite polynomial f, we know that its ***filled Julia set*** $K(f)$, besides being connected, is locally connected. Call a periodic orbit that contains a critical point a ***critical cycle***. In the postcritically finite setting, a periodic orbit belongs to the Fatou set $F(f)$ if and only if it is a critical cycle (for details we refer to [**M4**, Corollary 14.5]).

In this postcritically finite case, the polynomial f when restricted to the interior of $K(f)$ (which happens to be nonempty only when there exists a critical cycle) maps each bounded Fatou component —always simply connected by the maximum modulus principle— onto some other as a branched covering map. Furthermore, all of them are eventually periodic (see [**M4**, Theorem 16.4]). And also, each component can be uniformized so that in local charts f reads $z \mapsto z^m$ for some $m \geq 1$ (see [**M4**, Theorem 9.1]). More is true. If U is a periodic bounded Fatou component, then the first return map is conjugate to $z \mapsto z^k$, this time with $k \geq 2$. In particular, loops of components are in perfect correspondence with critical cycles. Also, in each component there is a unique point which eventually maps to a critical point (precisely the one marked as 0 in local coordinates), its ***center***.

It is well known (see for instance [**DH1**, Corollary VII.4.2]) that given a degree $d \geq 2$ postcritically finite polynomial f, for any $z \in K(f)$ the sets $K(f) - \{z\}$ and $J(f) - \{z\}$ consist both of a finite number of connected components. In this way, the filled Julia set can be thought of as arranged in a tree-like fashion.

To get rid of inessentials, we pick a finite invariant set M containing all critical points. Within $K(f)$ we interconnect M by arcs subject to the extra condition that when a Fatou component is met, then this intersection consists of radial segments in the associated coordinate. Douady and Hubbard proved that this construction defines a finite topological tree $T(M)$ when M together with the intrinsic branching points are considered vertices.

The vertex dynamics is invariant and carries the endpoints of any edge to distinct elements, so that it can be extended to a function from $T(M)$ to itself which is one to one on each edge and is isotopic to f, the original map. We also keep record of the local degree at every vertex v as $d(v)$. In addition, if three or more edges meet at a vertex, then their cyclic order should be remembered. In other words, we specify how this tree is embedded in the complex plane, again, up to isotopy.

Unfortunately, this data alone is not enough to determine the affine conjugacy class of f. However, if we append enough information to recover the inverse tree, then different postcritically finite polynomials yield different structures. To formally deal with this condition we introduce angles around vertices (this is to be credited again to [**DH1**]). In what follows we measure angles in turns, so that 1 degree measures 1/360 of a turn. At the center of a component the angle between edges is measured using the local chart. Near Julia vertices, where m components of $K(f)$ intersect, the angle is naturally defined as a multiple of $1/m$.

These angles satisfy two obvious conditions. First, they are compatible with the embedding of the tree. That is, as we go around a vertex in the positive direction, the successive angles are between zero and one, and add up to $+1$. Second, they

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satisfy the identity

$$\angle_{f(v)}(f(e), f(e')) = d(v)\angle_v(e, e') \mod 1,$$

where $d(v)$ is the local degree at v, and e, e' are edges incident at v. When this further structure is provided, we are in front of a ***Hubbard tree***.

Abstract Hubbard Trees. Now we move in reverse: we start with an abstract dynamical tree and we reconstruct the appropriate postcritically finite polynomial.

An ***angled tree*** H is a finite simplicial tree together with a function $e, e' \mapsto \angle_v(e, e') \in \mathbb{Q}/\mathbb{Z}$ which assigns a rational number modulo 1 to each pair of edges e, e' incident at a vertex v. The ***angle*** $\angle_v(e, e')$ is skew symmetric with $\angle_v(e, e') = 0$ if and only if $e = e'$, and satisfies $\angle_v(e, e'') = \angle_v(e, e') + \angle_v(e', e'')$ whenever the three edges meet at v. This angle function determines a preferred isotopy class of embeddings of H into $\mathbb{C}$.

Let V be the set of vertices in H. We specify a ***vertex dynamics*** $f : V \to V$ subject to $f(v) \neq f(v')$ whenever v, v' are end-points of a common edge e. We consider also a ***local degree*** $d : V \to \{1, 2, \dots\}$. We require that the ***total degree*** $d_H = 1 + \sum_{v \in V}(d(v) - 1)$ must be greater than 1. By definition a vertex is ***critical*** if $d(v) > 1$ and non-critical otherwise. The ***critical set*** is thus non-empty.

We require f and the degree d to be related to the angles as follows. Extend f to a map $f : H \to H$ that carries each edge homeomorphically onto the shortest path joining the images of its endpoints. We then need $\angle_{f(v)}(f(e), f(e')) = d(v)\angle_v(e, e')$ whenever e, e' are incident at v (so that $f(e)$, $f(e')$ intersect at the vertex $f(v)$ where the angle is measured).

A vertex v is ***periodic*** if $f^{\circ k}(v) = v$ for some $k \geq 1$. The orbit of a periodic critical point is a ***critical cycle***. A vertex is of ***Fatou type*** if it eventually maps to a critical cycle; else, it is of ***Julia type*** or a ***Julia vertex***.

The distance $dist_H(v, v')$ between vertices in H counts the number of edges in the shortest path joining v to v'. We call H ***expanding*** if for every edge e whose endpoints v, v' are Julia vertices there is $n \geq 1$ for which we have

$$dist_H(f^{\circ n}(v), f^{\circ n}(v')) > 1.$$

(In practice this property must be tested only for adjacent Julia vertices.)

Angles at Julia vertices are rather artificial, so it is better to normalize them. If m edges $e_1, \dots, e_m$ meet at a periodic Julia vertex v, then each $\angle_v(e_i, e_j)$ should be a multiple of $1/m$. (Therefore, angles around a periodic Julia vertex convey no information beyond the cyclic order of the edges.) An angled tree that satisfies this condition around all periodic Julia vertices is said to be ***normalized***.

By an ***abstract Hubbard tree*** —or simply a Hubbard tree— we mean a normalized angled tree that obeys the expanding condition. The basic existence and uniqueness theorem is stated now as follows.

THEOREM A.1 (Poirier [**P**]). *A normalized dynamical angled tree can be realized as the tree associated to a postcritically finite polynomial if and only if it is expanding, or in other words if and only if it is an abstract Hubbard tree. Such a realization is unique up to affine conjugation.*

Note that there many cases where we can apply this result directly: any tree which has no adjacent Julia vertices is trivially expanding. For instance, a star-shaped dynamical tree in which a critical cycle pivots around a fixed vertex can always be realized.

Now we are ready to realize a given reduced scheme $\overline{S} = (|\overline{S}|,\, F,\, \mathbf{w})$ and settle the existence of a postcritically finite polynomial f of degree $d(\overline{S}) = \mathbf{w}(\overline{S}) + 1$ whose associated reduced scheme $\overline{S}(f)$ is isomorphic to $\overline{S}$.

Theorem A.2. *Every reduced scheme can be realized by a postcritically finite polynomial.*

Proof. First we construct a non-reduced scheme S which reduces to $\overline{S}$. This is done by adding new vertices of weight zero in such a way that the associated graph $\Gamma(S)$ can be obtained from $\Gamma(\overline{S})$ by plotting an extra vertex within each old edge. (Compare Figures 11 and 12, where the new vertices are indicated by small circles.) More explicitly, starting with a reduced scheme $\overline{S}$ with associated map $\overline{F} : |\overline{S}| \to |\overline{S}|$, construct a non-reduced scheme S with associated map $F : |S| \to |S|$, where $|\overline{S}| \subset |S|$, as follows. The difference set $|S| - |\overline{S}|$ is to consist of one vertex $s^\sharp$ for each $s \in |\overline{S}|$, and the map $F : |S| \to |S|$ is defined by

$$F(s) = s^\sharp \qquad \text{and} \qquad F(s^\sharp) = s', \qquad \text{where} \qquad s' = \overline{F}(s).$$

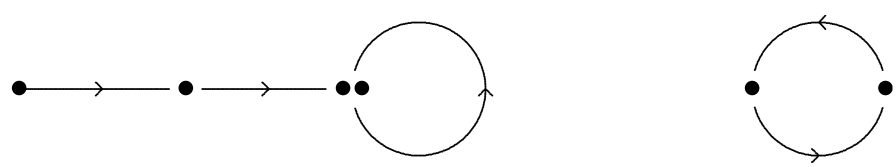

Figure 11. *A reduced scheme.*

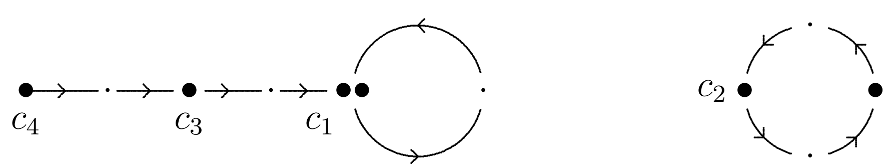

Figure 12. *The associated non reduced mapping scheme. From each cycle we pick a representative (c_1, c_2). All non-periodic critical points are also named (c_3, c_4).*

In simple words, $s^\sharp$ lies in the middle of s and $s' = \overline{F}(s)$, so that the main difference between F and $\overline{F}$ is that a vertex $s \in |\overline{S}|$ now takes an intermediate (artificial) step before reaching $s' = F(s) \in |\overline{S}|$. Formally, we have $F^{\circ 2} = \overline{F}$ when restricted to $|\overline{S}|$.

By its very definition, every vertex of the form $s^\sharp$ has s as its only preimage. From this construction it follows readily that S has $\overline{S}$ as its reduced scheme. This scheme is the one that we will bring to life with the help of a suitable expanding Hubbard tree.

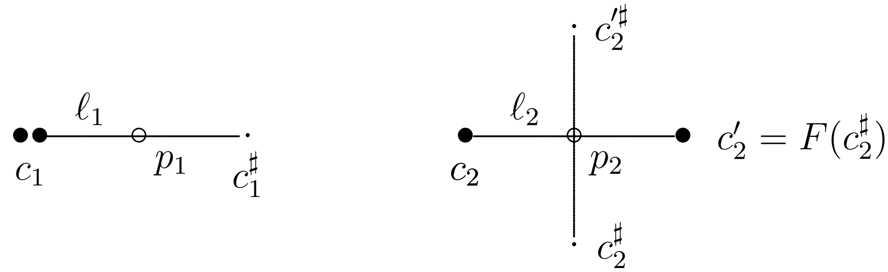

FIGURE 13. *The dynamical graphs corresponding to the two cycles in Figure 12.*

Let $\mathcal{C}_i$ be a cycle $s_0 \mapsto s_0^\sharp \mapsto s_1 \mapsto \cdots \mapsto s_{n-1}^\sharp \mapsto s_0$ in S. We join all these $2n$ vertices consecutively around a new vertex p_i in order to get a star-shaped symmetrical graph (compare Figure 13). Mapping p_i to itself, we have a Hubbard tree. (Here and elsewhere the degree $d(s) = \mathbf{w}(s) + 1$ is copied from the scheme.) All these s_k and $s_k^\sharp$ belong to a critical cycle and as such are of Fatou type. Hence, the dynamics in this graph is expanding since there is only one Julia type vertex in sight (the fixed point p_i). For future reference we pick a critical vertex in the loop (for instance s_0, which is critical because it belongs to $|\overline{S}|$ where $\overline{S}$ is reduced) and call it c_i. Also, the edge between c_i and p_i will be referred to as ℓ_i.

FIGURE 14. *The remaining non-cyclic edges.*

Let $c_1, \ldots, c_m$ be the vertices lying within a critical cycle, as numbered above. We now have to perform an extra auxiliary construction. Let

$$c_{m+1} = s_{m+1}, \ldots, c_{m+r} = s_{m+r} \in \overline{|S|}$$

be the non-periodic critical vertices, i.e, those outside a critical cycle. For $i = m+1, \ldots, m+r$ consider segments ℓ_i between $c_i = s_i$ and $s_i^\sharp$ (compare Figure 14).

Now we are ready to construct the Hubbard tree. For $i = 1, \cdots, m+r$, add a segment e_i at s_i making an angle of $1/d(s_i)$ units with ℓ_i. (This is to guarantee a complete folding at s_i when we iterate.) Merge the e_i's at a new vertex q making a uniform angle of $1/(m+r)$ between consecutive edges (the order here is irrelevant). Map the Julia vertex q to itself (compare Figure 15). Since there are no adjacent Julia vertices, we have an expanding Hubbard tree. The essentially unique postcritically finite polynomial that realizes this Hubbard tree, whose existence is guaranteed by Theorem A.1, clearly has the required reduced scheme $\overline{S}$. $\square$

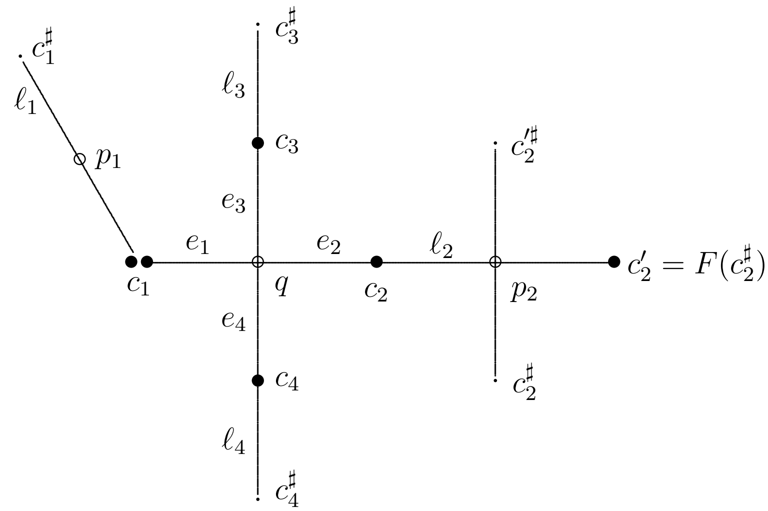

FIGURE 15. *Assembling the pieces together. Notice that at* c_1 *the angle should be* $1/3 \pmod 1$*. By construction, the Julia type fixed points* $p_1,\ q,\ p_2$ *(represented here by circles) have rotation numbers* $1/2,\ 1,\ 1/4$ *respectively. Also recall that* F *maps* $c_4^\sharp = F(c_4)$ *to* c_3 *and* $c_3^\sharp = F(c_3)$ *to* c_1.

Appendix B. Census of Reduced Schemes

In this appendix, all schemes are to be reduced. The object is to count the number $N(\mathbf{w})$ of distinct isomorphism classes of schemes of weight $\mathbf{w}(S) = \mathbf{w}$ for each small value of $\mathbf{w}$. The counting process can be broken down into a number of smaller steps as follows.

(a) Every scheme is uniquely a disjoint sum of connected schemes. If $S_1,\ S_2,\ S_3,\ \ldots$ is a list of all connected schemes, then every scheme can be expressed uniquely as a sum

$$S \;=\; S_{i(1)} + S_{i(2)} + \cdots + S_{i(k)}\;,$$

where k is the number of connected components, and where

$$i(1) \;\le\; i(2) \;\le\; \cdots \;\le\; i(k)\;.$$

The total weight of such a sum is $\mathbf{w}(S) = \sum_j \mathbf{w}(S_{i(j)})$. Thus, in order to compute the total number $N(\mathbf{w})$ of schemes of weight $\mathbf{w}$, it suffices to know the smaller number $N_c(\mathbf{w}')$ of connected schemes of weight $\mathbf{w}'$, for every $\mathbf{w}' \le \mathbf{w}$.

DEFINITION B.1. By a ***weighted tree*** will be meant an acyclic simplicial complex of dimension ≤ 1 with a preferred ***root*** vertex, together with a weight function which assigns a positive integer to each non-root vertex. By definition, the root vertex always has weight zero. The ***trivial*** weighted tree consists of the root vertex alone, with no edges.

(b) Every connected scheme S consists of a central cycle C of weight $\mathbf{w}(C) \ge 1$, together with a (possibly trivial) weighted tree $T(s)$ which is pasted

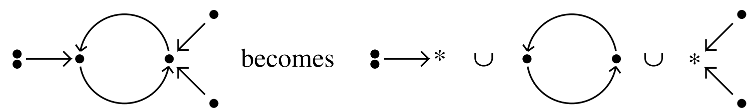

FIGURE 16. *Splitting off the trees from a scheme of weight* $\mathbf{w}(S) = 6$.

onto each vertex $s \in |C|$. Here the root point of $T(s)$ is to be identified with s. (Compare Figure 16, where each root point is represented by the symbol $*$.) Thus

$$\mathbf{w}(S) \;=\; \mathbf{w}(C) + \sum_{s\in|C|} \mathbf{w}\Big(T(s)\Big)\,, \qquad \text{where} \qquad \mathbf{w}(T) \;=\; \sum_{t\in|T|,\ t\neq *} \mathbf{w}(t)\,.$$

Note that the cycle C can be economically described by a symbol of the form $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ which is well defined up to cyclic permutation. Here the $\mathbf{w}_i$ are positive integers with sum $\mathbf{w}(C)$. Each $\mathbf{w}_i$ corresponds to a vertex s_i of weight $\mathbf{w}_i$ which maps to s_{i+1}, where the subscript i varies over $\mathbb{Z}/n$.

(c) By a ***trunk*** of a tree will be meant an edge incident to the root point. Thus every non-trivial tree has at least one trunk. Let $T_1, T_2, T_3, \ldots$ be a list of all trees with only one trunk. Then a tree with $k \geq 2$ trunks is isomorphic to a unique wedge sum

$$T \;=\; T_{i(1)} \vee \cdots \vee T_{i(k)} \tag{B.1}$$

of non-trivial trees pasted together at the root point, where $i(1) \leq \cdots \leq i(k)$. Just as in paragraph **(a)** above, the total weight is the sum $\mathbf{w}(T) = \sum_j \mathbf{w}\big(T_{i(j)}\big)$. (Remember that the weight of the root point is always zero.)

(d) If $N_{\rm tree}(\mathbf{w})$ is the number of distinct trees of weight $\mathbf{w}$, and $N_1(\mathbf{w})$ is the number of such trees with only one trunk, then

$$N_1(\mathbf{w}) \;=\; N_{\rm tree}(0) + N_{\rm tree}(1) + \cdots + N_{\rm tree}(\mathbf{w}-1)\,, \tag{B.2}$$

In fact if T is any tree with just one trunk $T_0 \subset T$, then by collapsing T_0 to a point we obtain a tree T/T_0 with weight

$$\mathbf{w}(T/T_0) \;=\; \mathbf{w}(T) \;-\; \mathbf{w}(T_0) \;<\; \mathbf{w}(T)\,.$$

Conversely, T can be reconstructed by pasting T/T_0 onto T_0. The identity (B.2) follows easily. Note that $N_{\rm tree}(0) = 1$, since there is a unique (trivial) tree of weight zero.

$\mathbf{w}$	0	1	2	3	4	5
$N_1(\mathbf{w})$	0	1	2	5	13	37
$N_{\rm tree}(\mathbf{w})$	1	1	3	8	24	71

TABLE 2. *Numbers of trees with given total weight* $\mathbf{w}$.

The values for $\mathbf{w} \leq 5$ are shown in Table 2. This table can be constructed inductively as follows. Suppose that we know the values $N_{\rm tree}(\mathbf{w}')$ for $\mathbf{w}' < \mathbf{w}$. Then $N_1(\mathbf{w})$ can be computed immediately from equation (B.2). On the other hand, if we know $N_1(\mathbf{w}')$ for all $\mathbf{w}' \leq \mathbf{w}$, then $N_{\rm tree}(\mathbf{w})$ can be computed as follows.

Note that any wedge sum expression (B.1) gives rise to a partition of the total weight $\mathbf{w}$, that is a sequence of positive integers which can be ordered so that

$$\mathbf{w}(T_{i(1)}) \;\le\; \mathbf{w}(T_{i(2)}) \;\le\; \cdots \;\le\; \mathbf{w}(T_{i(k)}),$$

with sum equal to $\mathbf{w}$. First consider the special case where all k of these wedge summands have the same weight $\mathbf{w}_0 = \mathbf{w}/k$. Then there are $N_1(\mathbf{w}_0)$ possible choices for each of these summands, where their order doesn't matter. The total number of possibilities in this case is equal to the binomial coefficient

$$\binom{N_1(\mathbf{w}_0)+k-1}{k}\,. \tag{B.3}$$

To see this, let a_h be the number of copies of the $h^{\rm th}$ tree in this k-fold wedge sum, so that $a_h \ge 0$ with $a_1+a_2+\cdots+a_{N_1(\mathbf{w}_0)} = k$. Then the partial sums $a_1+a_2+\cdots+a_i+i$ with $1\le i<N_1(\mathbf{w}_0)$ can be any increasing sequence of $N_1(\mathbf{w}_0)-1$ distinct integers between one and $N_1(\mathbf{w}_0)+k-1$. Hence the number of possibilities is given by the binomial coefficient (B.3).

More generally, suppose that for each n between 1 and $\mathbf{w}$ there are $k_n \ge 0$ summands of weight n, so that

$$\mathbf{w} \;=\; k_1 + 2\,k_2 + \cdots + \mathbf{w}\,k_{\mathbf{w}}\,.$$

Then the number of possibilities is equal to the product

$$\prod_{k_n>0}\binom{N_1(n)+k_n-1}{k_n}\,,$$

taken over all n with $k_n>0$. Thus, all together, the number of possibilities is given by

$$N_{\rm tree}(\mathbf{w}) \;=\; \sum_{\rm partitions}\;\prod_{k_n>0}\binom{N_1(n)+k_n-1}{k_n}\,, \tag{B.4}$$

to be summed over all partitions $\sum n\,k_n \;=\; \mathbf{w}$.

As an example, suppose that $N_1(\mathbf{w})$ is known for $\mathbf{w}\le 4$. Using the values shown in Table 2, since the integer 4 has 5 different partitions

$$1+1+1+1 \;=\; 1+1+2 \;=\; 1+3 \;=\; 2+2 \;=\; 4$$

(using a different notation for partitions), it follows that $N_{\rm tree}(4)$ can be expressed as a 5-fold sum

$$\binom{4}{4} \;+\; \binom{2}{2}\binom{2}{1} \;+\; \binom{1}{1}\binom{5}{1} \;+\; \binom{3}{2} \;+\; \binom{13}{1}\,,$$

yielding

$$N_{\rm tree}(4) \;=\; 1+2+5+3+13 \;=\; 24\,.$$

Other entries in Table 2 can be computed similarly.

The following table lists the number of connected schemes for each given value of the cyclic weight $\mathbf{w}(C)$ together with the tree weight $\sum_{s\in|C|}\mathbf{w}(T(s))$, within the range $\mathbf{w}(S) \;=\; \mathbf{w}(C)+\sum\mathbf{w}(T) \;\le\; 6$.

Rather than explaining each entry in this table, let me simply give a detailed explanation for the one typical entry which is underlined in the table, corresponding to cyclic weight $\mathbf{w}(C)=3$, tree weight $\sum\mathbf{w}(T)=2$, and hence total weight $3+2=5$. For this example, we need to know the numbers $N_{\rm tree}(1)=1$ and $N_{\rm tree}(2)=3$, and

$\sum \mathbf{w}(T) =$		0	1	2	3	4	5
$\mathbf{w}(C) =$	1	1	1	3	8	24	71
	2	2	2	7	19	62	
	3	3	4	$\underline{14}$	45		
	4	5	8	31			
	5	7	16				
	6	13					

Table 3. *Number of connected schemes S with given $\mathbf{w}(C)$ and $\sum \mathbf{w}(T)$.*

we need to study each of the three cyclic schemes of weight $\mathbf{w}(C) = 3$ separately. Here are the three cases, with notation as in **(b)** above.

- For the cycle (3), with a single vertex of weight 3, we can paste any one of the three trees of weight 2 onto the unique cyclic vertex, so we get a total of 3 possible schemes.
- For the cycle $(1, 2)$, we can paste a tree of weight 2 onto either one of the two vertices, yielding 6 distinct possibilities. But we can also paste a tree of weight one onto each vertex, yielding a 7-th possibility.
- For the cycle $(1, 1, 1)$, note that there is a cyclic group of symmetries. Again we can paste a tree of weight 2 onto any vertex, but because of the symmetries, it doesn't matter which vertex we choose, so there are three distinct possibilities. Similarly, we can paste a tree of weight one onto each of two vertices, yielding a 4-th possibility. (Again, because of the symmetries, it doesn't matter which two we choose.)

Thus, all together, we get $3 + 7 + 4 = 14$ distinct schemes, as listed in Table 3. Other entries in this table can be computed similarly.

The number $N_{\mathrm{c}}(\mathbf{w})$ of connected schemes of weight $\mathbf{w}$ can be obtained by adding entries along the diagonal $\mathbf{w}(C) + \sum \mathbf{w}(T) = \mathbf{w}$ in Table 3. For example

$$N_{\mathrm{c}}(5) \; = \; 7 + 8 + 14 + 19 + 24 \; = \; 72\,.$$

For $n \leq 6$, the total number $N_{\mathrm{c}}(\mathbf{w})$ of connected schemes of weight $\mathbf{w}$, computed in this way, is listed in the middle row of Table 4. The total number $N(\mathbf{w})$ of all schemes, connected or not, can then be computed by a formula completely analogous to (B.4) above, and is listed in the last row below (as well as in Table 1).

$\mathbf{w}$	1	2	3	4	5	6
$N_{\mathrm{c}}(\mathbf{w})$	1	3	8	24	72	238
$N(\mathbf{w})$	1	4	12	42	138	494

Table 4. *The total count.*

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